

Wall rational functions and Khrushchev's formula for orthogonal rational functions*

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Abstract

We prove that the Nevalinna-Pick algorithm provides different homeomorphisms between certain topological spaces of measures, analytic functions and sequences of complex numbers. This algorithm also yields a continued fraction expansion of every Schur function, whose approximants are identified. The approximants are quotients of rational functions which can be understood as the rational analogs of the Wall polynomials. The properties of these Wall rational functions and the corresponding approximants are studied. The above results permit us to obtain a Khrushchev's formula for orthogonal rational functions. An introduction to the convergence of the Wall approximants in the indeterminate case is also presented.

Keywords and phrases: Schur and Carathéodory functions; Nevalinna-Pick algorithm; orthogonal and Wall rational functions; Khrushchev's formula.

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1 Introduction

It is known that the Cayley transform provides a correspondence between Schur and Carathéodory functions. Besides, the integral representation of Carathéodory functions establishes a connection with finite positive Borel measures on the unit circle. On the other hand, the Schur algorithm associates with any Schur function the so called Schur parameters: a sequence in the open unit disk, the last point lying on the unit circle in the case of a terminating sequence. Indeed, the set of these complex sequences, the set of probability measures, the set of normalized Carathéodory functions and the set of Schur functions become homeomorphic under suitable topologies.

The homeomorphism with the sequences of Schur parameters yields a bi-continuous parametrization of the Schur functions or, alternatively, of the probability measures on the unit circle. The study of such a parametrization is important, not only for the theory of analytic functions, but also for the theory of continued fractions because the Schur algorithm is equivalent to a continued fraction expansion of Schur functions (hence, to a continued fraction for Carathéodory functions too). On the other hand, the parametrization of measures on the unit circle becomes specially significant for the associated orthogonal polynomials, since the Schur parameters are the coefficients of the corresponding recurrence relation. The orthogonal polynomials on the unit circle also provide the numerators and denominators of the approximants for the continued fraction expansion of the related Carathéodory function. A similar role for the case of Schur functions is played by the so called Wall polynomials, closely related to the orthogonal polynomials too.

Therefore, the above homeomorphisms permit us to connect problems concerning measures, orthogonal polynomials, continued fractions, analytic functions and complex sequences, so that one can translate results or choose the best context to work. A remarkable example of this is Krushchev's theory (see [13, 14]), which takes advantage of these connections to reach deep and impressive results on the referred matters. A key result in Khrushchev's theory is the so called Khrushchev's formula, obtained in [13] starting from the analysis of the Wall polynomials. This formula can be understood as the identification of the Schur functions of certain varying measures obtained by an orthogonal polynomial modification of the orthogonality measure.

The Schur algorithm is a characterization of Schur functions based on an iteration which evaluates each iterate at the origin. The Nevalinna-Pick algorithm, related to the interpolation of Schur functions, generalizes this

procedure evaluating each iterate at a different point of the open unit disk. Like the Schur algorithm, the Nevalinna-Pick generalization associates with any Schur function a similar sequence of parameters, but depending now on the choice of the evaluation points. The Nevalinna-Pick algorithm is also related to a rational generalization of the orthogonal polynomials on the unit circle: the orthogonal rational functions with prescribed poles outside the unit circle. It is known that these orthogonal rational functions are involved in alternative continued fraction expansions of Carathéodory functions. However, the corresponding continued fractions associated with Schur functions are not discussed in the literature. The related approximants should have as numerators and denominators certain rational functions depending on the evaluation points, which we will call Wall rational functions.

Finally we must comment a remarkable new phenomenon of the Nevalinna-Pick algorithm which does not appear in the Schur one: when the evaluation points approach to the unit circle quickly enough, an indeterminate case can appear, i.e., different Schur functions can have the same Nevalinna-Pick parameters. This causes important difficulties in the study of the convergence of the corresponding continued fraction, which now can have different limit points.

Once we have situated the context, we can understand the interest of our work, whose aims are:

- The analysis of the homeomorphisms related to the Nevalinna-Pick algorithm (Section 2).
- The study of the Wall rational functions and the corresponding continued fraction approximants of Schur functions (Section 3).
- The search for a Krushchev's formula for orthogonal rational functions (Section 4).
- An introduction to the analysis of the limit points of continued fractions for Schur functions in the indeterminate case (Section 5).

We will follow Khrushchev's approach to the polynomial case given in [13, 14]. As we will see, the approximants of a Schur function related to the Wall rational functions are the Schur functions corresponding to the approximants of the continued fraction for the related Carathéodory function. Hence, bearing in mind the homeomorphism between Schur and Carathéodory functions, the convergence of the Schur continued fraction is equivalent to the convergence of the Carathéodory continued fraction. Indeed, we will show

that the convergence of both continued fractions can be understood as a consequence of the asymptotics of the Nevalinna-Pick parameters corresponding to the related approximants. These convergence results are limited by the validity of the homeomorphism for the Nevalinna-Pick parametrization of the Schur functions, which is ensured in the determinate case.

The results about the Wall rational functions and the homeomorphisms related to the Nevalinna-Pick algorithm will be the main tools to prove a Khrushchev's formula for orthogonal rational functions. This will be the starting point of a "rational Khrushchev's theory" whose development will be given elsewhere. Nevertheless, a first application of Khrushchev's formula will appear in the study of the indeterminate case. The reason is that, contrary to the standard polynomial techniques, which usually can be extended only to the determinate rational case, the rational generalization of Khrushchev's formula always holds, providing an important tool for the study of the indeterminate case. Nevertheless, our approach to the indeterminate case will be only introductory, trying simply to show the variety of situations that can appear in the convergence of the related continued fractions. A more complete study of the indeterminate case deserves further investigations.

2 Nevalinna-Pick homeomorphisms

The results that we will prove here hold, not only for Schur functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, but also for Schur functions on the upper half plane $\mathbb{U} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, as can be seen using the Cayley transform. We will use a unified notation to present simultaneously the results in both situations, and when we want to distinguish between them we will write a left brace with the \mathbb{D} case in the first line and the \mathbb{U} case in the second one. For instance, in what follows we will use the notation

$$\mathbb{O} = \begin{cases} \mathbb{D}, \\ \mathbb{U}, \end{cases} \quad \partial\mathbb{O} = \begin{cases} \mathbb{T}, \\ \mathbb{R}, \end{cases} \quad \mathbb{O}^e = \overline{\mathbb{C}} \setminus \overline{\mathbb{O}},$$

where \overline{S} is the closure in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of a subset $S \subset \overline{\mathbb{C}}$.

Consider the transformations ζ_α , $\alpha \in \mathbb{O}$, given by

$$\zeta_\alpha = z_\alpha \frac{\varpi_\alpha^*}{\varpi_\alpha}, \quad z_\alpha = \begin{cases} -\frac{|\alpha|}{\alpha}, \\ \frac{|1+\alpha^2|}{1+\alpha^2}, \end{cases} \quad \varpi_\alpha(z) = \begin{cases} 1 - \overline{\alpha}z, \\ z - \overline{\alpha}, \end{cases} \quad \varpi_\alpha^*(z) = z - \alpha,$$

where we understand that $z_\alpha = 1$ for the particular value $\alpha = \alpha_0$ with

$$\alpha_0 = \begin{cases} 0, \\ i. \end{cases}$$

ζ_α is a homeomorphism of $\overline{\mathbb{C}}$ which maps \mathbb{O} , $\partial\mathbb{O}$ and \mathbb{O}^e onto \mathbb{D} , \mathbb{T} and $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ respectively.

A useful identity for ζ_α is

$$\zeta_\alpha(t) - \zeta_\alpha(z) = z_\alpha \frac{\varpi_\alpha(\alpha) \varpi_z^*(t)}{\varpi_\alpha(t) \varpi_\alpha(z)}. \quad (1)$$

Besides, if we define the substar operation on complex functions by

$$f_*(z) = \overline{f(\hat{z})}, \quad \hat{z} = \begin{cases} 1/\bar{z}, \\ \bar{z}, \end{cases}$$

then

$$\zeta_{\alpha*} = \zeta_{\hat{\alpha}} = 1/\zeta_\alpha. \quad (2)$$

The sets that will be involved in the homeomorphisms are

\mathfrak{P} = set of finite Borel measures on $\partial\mathbb{O}$, $\mathfrak{P}_0 = \{d\mu \in \mathfrak{P} : \mu_0 = \int d\mu = 1\}$,

$\mathfrak{C} = \{F \in \mathcal{H}(\mathbb{O}) : \operatorname{Re} F(z) > 0 \ \forall z \in \mathbb{O}\}$, $\mathfrak{C}_\alpha = \{F \in \mathfrak{C} : F(\alpha) = 1\}$,

$\mathfrak{B} = \{f \in \mathcal{H}(\mathbb{O}) : |f(z)| \leq 1 \ \forall z \in \mathbb{O}\}$,

$$\mathfrak{S} = \left\{ \gamma = (\gamma_n)_{n=0}^N : N \in \{0, 1, \dots, \infty\}, \gamma_n \in \begin{cases} \mathbb{D} & \text{if } n < N \\ \mathbb{T} & \text{if } n = N < \infty \end{cases} \right\},$$

where $\alpha \in \mathbb{O}$ and $\mathcal{H}(S)$ is the set of analytic functions on the subset $S \subset \mathbb{C}$. We will consider the topologies

set	topology	notation
\mathfrak{P}	*-weak convergence	$d\mu^k \xrightarrow{*} d\mu$
$\mathfrak{C}, \mathfrak{B}$	uniform convergence in compact subsets of \mathbb{O}	$f^k \rightrightarrows f$
\mathfrak{S}	pointwise convergence	$\gamma^k \rightarrow \gamma$

The elements of \mathfrak{B} and \mathfrak{C} are called Schur and Carathéodory functions respectively or, in short, S-functions and C-functions. We will assume that any Schur or Carathéodory function f is extended to $\partial\mathbb{D}$ by

$$f(z) = \begin{cases} \lim_{r \uparrow 1} f(rz), \\ \lim_{\epsilon \downarrow 0} \operatorname{Re} F(z + i\epsilon), \end{cases} \quad \text{a.e. } z \in \partial\mathbb{D},$$

since it is known that such limits exists a.e. on $\partial\mathbb{D}$.

The set of limit points of a sequence (x^k) in a topological space will be denoted $\operatorname{Lim} x^k$. We will use for the pointwise convergence in the space of complex sequences the same notation as in the case of \mathfrak{S} . Concerning the convergence of an arbitrary sequence (f^k) of complex functions, the notation

$$f^k \rightrightarrows f \text{ in } S$$

means that f^k converges uniformly to f in compact subsets of $S \subset \mathbb{C}$.

For convenience, when using α_0 as a subindex we will usually identify it with 0, thus $\mathfrak{C}_0 = \mathfrak{C}_{\alpha_0}$, $\zeta_0 = \zeta_{\alpha_0}$, etc. In particular,

$$\zeta_0(z) = \begin{cases} z, \\ \frac{z-i}{z+i}, \end{cases}$$

that is, ζ_0 is the identity in \mathbb{D} or the Cayley transform in \mathbb{U} .

2.1 Measures, C-functions and S-functions

Concerning the relation between measures and C-functions, it is known that \mathfrak{C} is homeomorphic to $\mathfrak{P} \times \mathbb{R}$ through

$$\begin{aligned} \mathfrak{P} \times \mathbb{R} &\longrightarrow \mathfrak{C} \\ (d\mu, c) &\rightarrow F(z; d\mu) + ic \end{aligned} \quad (3)$$

$$F(z; d\mu) = \int D(t, z) d\mu(t), \quad D(t, z) = \frac{\zeta_0(t) + \zeta_0(z)}{\zeta_0(t) - \zeta_0(z)} = \begin{cases} \frac{t+z}{t-z}, \\ \frac{1}{i} \frac{1+tz}{t-z}. \end{cases}$$

In other words, \mathfrak{P} is homeomorphic to the set of C-functions with the form $F(z; d\mu)$, which are exactly the C-functions real valued at the origin since $F(0; d\mu) = \mu_0$. We have also the induced homeomorphism

$$\begin{aligned} \mathcal{C}_0: \mathfrak{P}_0 &\longrightarrow \mathfrak{C}_0 \\ d\mu &\rightarrow F(z; d\mu) \end{aligned} \quad (4)$$

If μ' is the derivative of $d\mu$ with respect to the Lebesgue measure, it is known that

$$\operatorname{Re} F(z; d\mu) = \mu'(z) \quad \text{a.e. } z \in \partial\mathbb{O}. \quad (5)$$

Some identities for $D(t, z)$ will be useful later. Let us start defining

$$D_R(t, z) = \frac{1}{2}(D(t, z) + D_*(t, z)), \quad D_I(t, z) = \frac{1}{2i}(D(t, z) - D_*(t, z)),$$

where the substar operation on $D(\cdot, \cdot)$ is taken always on the first argument. Then, $D_R(t, z) = \operatorname{Re} D(t, z)$ and $D_I(t, z) = \operatorname{Im} D(t, z)$ for $t \in \partial\mathbb{O}$. Using properties (1) and (2) for ζ_0 we find that

$$D_*(t, z) = -D(t, \hat{z})$$

and

$$D(t, z) - D(t, \alpha) = 2 \frac{\varpi_0(t) \varpi_0^*(t) \varpi_\alpha^*(z)}{\varpi_0(\alpha_0) \varpi_z^*(t) \varpi_\alpha^*(t)}. \quad (6)$$

Taking the substar operation with respect to t on (6) and changing z by \hat{z} we get

$$D(t, z) + D_*(t, \alpha) = 2 \frac{\varpi_0(t) \varpi_0^*(t) \varpi_\alpha(z)}{\varpi_0(\alpha_0) \varpi_z^*(t) \varpi_\alpha(t)}, \quad (7)$$

which gives

$$D_R(t, z) = \frac{\varpi_z(z) \varpi_0(t) \varpi_0^*(t)}{\varpi_0(\alpha_0) \varpi_z(t) \varpi_z^*(t)}. \quad (8)$$

In particular,

$$D_R(t, z) = \operatorname{Re} D(t, z) = \frac{\varpi_z(z)}{\varpi_0(\alpha_0)} \left| \frac{\varpi_0(t)}{\varpi_z(t)} \right|^2 = \begin{cases} \frac{1-|z|^2}{|t-z|^2}, \\ \frac{\operatorname{Im} z (1+t^2)}{|t-z|^2}, \end{cases} \quad t \in \partial\mathbb{O}. \quad (9)$$

While the homeomorphism (4) is the relevant one for the polynomial setting, its generalization to \mathfrak{C}_α for any $\alpha \in \mathbb{O}$ will be important for the rational case. To understand this generalization notice that

$$\begin{aligned} \mathfrak{P} &\longrightarrow \mathfrak{P} \\ d\mu &\rightarrow \frac{d\mu(\cdot)}{D_R(\cdot, \alpha)} \end{aligned}$$

is a homeomorphism since $D_R(t, \alpha)$ is positive and continuous for all $t \in \partial\mathbb{D}$. Composing it with (3) shows that

$$\begin{aligned} \mathfrak{P} \times \mathbb{R} &\longrightarrow \mathfrak{C} \\ (d\mu, c) &\rightarrow F\left(z; \frac{d\mu(\cdot)}{D_R(\cdot, \alpha)}\right) + ic \end{aligned}$$

is a homeomorphism too. It is straightforward to see that this homomorphism induces the following one

$$\begin{aligned} \mathcal{C}_\alpha: \mathfrak{P}_0 &\longrightarrow \mathfrak{C}_\alpha \\ d\mu &\rightarrow F_\alpha(z; d\mu) \end{aligned} \tag{10}$$

where $F_\alpha(z; d\mu)$ is defined for any $d\mu \in \mathfrak{P}$ and any $\alpha \in \mathbb{D}$ by

$$F_\alpha(z; d\mu) = F\left(z; \frac{d\mu(\cdot)}{D_R(\cdot, \alpha)}\right) + ic_\alpha(d\mu), \quad c_\alpha(d\mu) = - \int \frac{D_I(t, \alpha)}{D_R(t, \alpha)} d\mu(t).$$

We will say that $F_\alpha(z; d\mu)$ is the α -C-function of $d\mu$. $F_\alpha(\alpha; d\mu) = \mu_0$, thus, a C-function has the form $F_\alpha(z; d\mu)$ for some $d\mu \in \mathfrak{P}$ iff it is real at α , and the set of these C-functions is homeomorphic to \mathfrak{P} .

A stronger convergence property than the one given by the homeomorphism (10) holds. To prove it we will use an explicit relation between $F_\alpha(z; d\mu)$ and $F(z; d\mu)$. Although such a relation was obtained in [7, Lemmas 6.2.2 and 6.2.3], we present here a more concise proof which, at the same time, unifies the discussion for measures on the unit circle and the real line.

Proposition 2.1. *For any $d\mu \in \mathfrak{P}$ and any $\alpha \in \mathbb{D}$,*

$$F(z; d\mu) = D_R(z, \alpha) F_\alpha(z; d\mu) - i\mu_0 D_I(z, \alpha).$$

Proof. From (6) and (7) we find that

$$\begin{aligned} D(t, z) - iD_I(t, \alpha) &= \frac{1}{2}(D(t, z) - D(t, \alpha)) + \frac{1}{2}(D(t, z) - D_*(t, \alpha)) = \\ &= \frac{\varpi_0(t) \varpi_0^*(t)}{\varpi_0(\alpha_0) \varpi_z^*(t)} \left(\frac{\varpi_\alpha^*(z)}{\varpi_\alpha^*(t)} + \frac{\varpi_\alpha(z)}{\varpi_\alpha(t)} \right), \end{aligned}$$

which combined with (8) gives

$$\frac{D(t, z)}{D_R(t, \alpha)} - i \frac{D_I(t, \alpha)}{D_R(t, \alpha)} = \frac{\varpi_\alpha(t) \varpi_\alpha^*(z) + \varpi_\alpha^*(t) \varpi_\alpha(z)}{\varpi_\alpha(\alpha) (t - z)}.$$

The above function, as well as $D(t, z)$, are antisymmetric under the exchange of t and z . Hence,

$$\frac{D(t, z)}{D_R(t, \alpha)} - i \frac{D_I(t, \alpha)}{D_R(t, \alpha)} = \frac{D(t, z)}{D_R(z, \alpha)} + i \frac{D_I(z, \alpha)}{D_R(z, \alpha)},$$

which, integrated with respect to $d\mu(t)$, finally yields the result. \square

The above relation permits us to obtain a convergence property for sequences of C-functions normalized at different points. Notice that, as a consequence of the maximum modulus principle, if $F^k, F \in \mathcal{H}(\mathbb{O})$ and $F^k \rightrightarrows F$ in $\mathbb{O} \setminus K$, K a compact subset of \mathbb{O} , then $F^k \rightrightarrows F$ in \mathbb{O} . If, besides, $F^k \in \mathfrak{C}_{\alpha_k}$ with $\alpha_k \in \mathbb{O}$ such that $\alpha_k \rightarrow \alpha \in \mathbb{O}$, then $F \in \mathfrak{C}_\alpha$, so we can suppose in this situation that $F^k(z) = F_{\alpha_k}(z; d\mu^k)$ and $F(z) = F_\alpha(z; d\mu)$ for some probability measures $d\mu^k, d\mu$.

Theorem 2.2. *Let (α_k) be a sequence in \mathbb{O} and $(d\mu^k)$ a sequence in \mathfrak{P} . If $\alpha_k \rightarrow \alpha \in \mathbb{O}$, then*

$$F_{\alpha_k}(z; d\mu^k) \rightrightarrows F_\alpha(z; d\mu) \iff d\mu^k \xrightarrow{*} d\mu.$$

Proof. It suffices to prove $F_{\alpha_k}(z; d\mu^k) \rightrightarrows F_\alpha(z; d\mu) \Leftrightarrow F(z; d\mu^k) \rightrightarrows F(z; d\mu)$. If $F_{\alpha_k}(z; d\mu^k) \rightrightarrows F_\alpha(z; d\mu)$, then $\mu_0^k = F_{\alpha_k}(\alpha_k; d\mu^k) \rightarrow \mu_0 = F_\alpha(\alpha; d\mu)$. From this result, Proposition 2.1, and the fact that $D_R(\cdot, \alpha_k) \rightrightarrows D_R(\cdot, \alpha)$, $D_I(\cdot, \alpha_k) \rightrightarrows D_I(\cdot, \alpha)$ in $\mathbb{O} \setminus \{\alpha\}$, we conclude that $F(z; d\mu^k) \rightrightarrows F(z; d\mu)$ in $\mathbb{O} \setminus \{\alpha\}$, so it holds in \mathbb{O} too. A similar reasoning proves the opposite implication, bearing in mind that $1/D_R(\cdot, \alpha_k) \rightrightarrows 1/D_R(\cdot, \alpha)$ in $\mathbb{O} \setminus \{\alpha_0\}$. \square

As for the connection with Schur functions, the relations

$$f = \frac{1}{\zeta_\alpha} \frac{F - 1}{F + 1}, \quad F = \frac{1 + \zeta_\alpha f}{1 - \zeta_\alpha f},$$

define a one to one mapping between C-functions $F \in \mathfrak{C}_\alpha$ and S-functions $f \in \mathfrak{B}$. Moreover, for any $\alpha \in \mathbb{O}$, the bijection

$$\mathcal{B}_\alpha: \begin{array}{l} \mathfrak{C}_\alpha \rightarrow \mathfrak{B} \\ F \rightarrow f \end{array} \quad (11)$$

is also a homeomorphism, as the following more general property shows.

Theorem 2.3. Let $(\alpha_k), (\beta_k)$ be two sequences compactly included in \mathbb{O} , $F^k \in \mathfrak{C}_{\alpha_k}, G^k \in \mathfrak{C}_{\beta_k}$ and $f^k = \mathcal{B}_{\alpha_k}(F^k), g^k = \mathcal{B}_{\beta_k}(G^k)$. If $\alpha_k - \beta_k \rightarrow 0$, then

$$F^k - G^k \rightrightarrows 0 \iff f^k - g^k \rightrightarrows 0.$$

Proof. The result follows easily from the identities

$$\begin{aligned} f^k - g^k &= \frac{1}{\zeta_{\alpha_k}} \left[(\zeta_{\beta_k} - \zeta_{\alpha_k})g^k + 2 \frac{F^k - G^k}{(F^k + 1)(G^k + 1)} \right], \\ F^k - G^k &= 2 \frac{(\zeta_{\alpha_k} - \zeta_{\beta_k})f^k + \zeta_{\beta_k}(f^k - g^k)}{(1 - \zeta_{\alpha_k}f^k)(1 - \zeta_{\beta_k}g^k)}, \end{aligned}$$

which give in \mathbb{O} the inequalities

$$|f^k - g^k| \leq \frac{|\zeta_{\alpha_k} - \zeta_{\beta_k}| + 2|F^k - G^k|}{|\zeta_{\alpha_k}|}, \quad |F^k - G^k| \leq 2 \frac{|\zeta_{\alpha_k} - \zeta_{\beta_k}| + |f^k - g^k|}{(1 - |\zeta_{\alpha_k}|)(1 - |\zeta_{\beta_k}|)}.$$

When $(\alpha_k), (\beta_k)$ are compactly included in \mathbb{O} and $\alpha_k - \beta_k \rightarrow 0$, the above inequalities prove that $f^k - g^k \rightrightarrows 0$ implies $F^k - G^k \rightrightarrows 0$, while $F^k - G^k \rightrightarrows 0$ implies $f^k - g^k \rightrightarrows 0$ in $\mathbb{O} \setminus \text{Lim } \alpha_k$, thus in \mathbb{O} because $\text{Lim } \alpha_k$ is a compact subset of \mathbb{O} . \square

Given $d\mu \in \mathfrak{P}_0$, the S-function $f_\alpha(z; d\mu) = \mathcal{B}_\alpha(F_\alpha(z; d\mu))$, will be called the α -S-function of $d\mu$. The relation between $F_\alpha(z; d\mu)$ and $F(z; d\mu)$ provides an explicit expression of the α -S-function $f_\alpha(z; d\mu)$ of $d\mu$ in terms of its α_0 -S-function $f(z; d\mu) = f_{\alpha_0}(z; d\mu)$.

Proposition 2.4. Let $d\mu \in \mathfrak{P}_0$ and $\alpha \in \mathbb{O}$. Denoting $f(z) = f(z; d\mu)$ and $f_\alpha(z) = f_\alpha(z; d\mu)$,

$$f_\alpha = -\frac{\zeta_0(\alpha)}{|\zeta_0(\alpha)|} \frac{f - \overline{\zeta_0(\alpha)}}{1 - \zeta_0(\alpha)f}, \quad f = -\frac{|\zeta_0(\alpha)|}{\zeta_0(\alpha)} \frac{f_\alpha - |\zeta_0(\alpha)|}{1 - |\zeta_0(\alpha)|f_\alpha}.$$

Proof. From Proposition 2.1 we find that

$$f_\alpha = \frac{1}{\zeta_\alpha} \frac{(1 - D_{\alpha*}) + (1 + D_{\alpha*})\zeta_0 f}{(1 + D_\alpha) + (1 - D_\alpha)\zeta_0 f}, \quad D_\alpha(z) = D(z, \alpha).$$

Besides, a direct calculation using the properties of ζ_0 gives

$$\frac{1 - D_{\alpha*}}{1 + D_\alpha} = |\zeta_0(\alpha)| \zeta_\alpha, \quad \frac{1 + D_{\alpha*}}{1 - D_\alpha} = \frac{\zeta_\alpha}{|\zeta_0(\alpha)|}, \quad \frac{1 + D_\alpha}{1 - D_\alpha} = -\frac{\zeta_0}{\zeta_0(\alpha)}.$$

From the first two identities we obtain

$$f_\alpha = \frac{|\zeta_0(\alpha)|(1 + D_\alpha) + |\zeta_0(\alpha)|^{-1}(1 - D_\alpha)\zeta_0 f}{(1 + D_\alpha) + (1 - D_\alpha)\zeta_0 f},$$

and, then, the last of the three identities yields the result. \square

In what follows, we will refer to $F(z; d\mu)$ and $f(z; d\mu)$ as the C-function and S-function of $d\mu$ respectively.

Proposition 2.4 can be combined with Theorem 2.3 to give the following general equivalences.

Theorem 2.5. *Let $(\alpha_k), (\beta_k)$ be two sequences compactly included in \mathbb{O} and $(d\mu^k), (d\nu^k)$ two sequences in \mathfrak{P}_0 . If $\alpha_k - \beta_k \rightarrow 0$, then*

$$\begin{aligned} F_{\alpha_k}(z; d\mu^k) - F_{\beta_k}(z; d\nu^k) \rightrightarrows 0 &\iff f_{\alpha_k}(z; d\mu^k) - f_{\beta_k}(z; d\nu^k) \rightrightarrows 0 \iff \\ &\iff d\mu^k - d\nu^k \xrightarrow{*} 0. \end{aligned}$$

Proof. Suppose that $(\alpha_k), (\beta_k)$ are compactly included in \mathbb{O} and $\alpha_k - \beta_k \rightarrow 0$. Then, Theorem 2.3 ensures the first equivalence. With the help of Proposition 2.4 we find that $f_{\alpha_k}(z; d\mu^k) - f_{\beta_k}(z; d\nu^k) \rightrightarrows 0$ iff $f(z; d\mu^k) - f(z; d\nu^k) \rightrightarrows 0$. Applying again Theorem 2.3 we conclude that $f(z; d\mu^k) - f(z; d\nu^k) \rightrightarrows 0$ iff $F(z; d\mu^k) - F(z; d\nu^k) \rightrightarrows 0$. This last condition is equivalent to $d\mu^k - d\nu^k \xrightarrow{*} 0$ because $F(z; d\mu^k) - F(z; d\nu^k) = F(z; d\mu^k - d\nu^k)$. \square

Example 2.6. Let us define for any $\alpha \in \mathbb{O}$ the measure

$$dm_\alpha(t) = \frac{\varpi_\alpha(\alpha)}{\varpi_\alpha(t) \varpi_\alpha^*(t)} \frac{dt}{2\pi i} = \begin{cases} \frac{1-|\alpha|^2}{|t-\alpha|^2} \frac{dt}{2\pi i}, & t \in \partial\mathbb{O}. \\ \frac{\operatorname{Im} \alpha}{|t-\alpha|^2} \frac{dt}{\pi}, & \end{cases}$$

In particular, $dm = dm_{\alpha_0}$ is the Lebesgue measure in \mathbb{T} or its Cayley transform in \mathbb{R} , i.e.,

$$dm(t) = \begin{cases} \frac{dt}{2\pi i t}, & t \in \partial\mathbb{O}. \\ \frac{dt}{\pi(1+t^2)}, & \end{cases}$$

Therefore, $F(z; dm) = 1$, so $f(z; dm) = 0$ and, from Proposition 2.4,

$$f_\alpha(z; dm) = |\zeta_0(\alpha)|, \quad F_\alpha(z; dm) = \frac{1 + |\zeta_0(\alpha)| \zeta_\alpha(z)}{1 - |\zeta_0(\alpha)| \zeta_\alpha(z)}.$$

In the general case, from (8) we get $dm_\alpha(t) = D_R(t, \alpha) dm(t)$, thus we have the equality

$$F\left(z; \frac{dm_\alpha(\cdot)}{D_R(\cdot, \alpha)}\right) = F(z; dm) = 1,$$

which gives

$$\int dm_\alpha = \operatorname{Re} F\left(z; \frac{dm_\alpha(\cdot)}{D_R(\cdot, \alpha)}\right) = 1, \quad c_\alpha(dm_\alpha) = -\operatorname{Im} F\left(z; \frac{dm_\alpha(\cdot)}{D_R(\cdot, \alpha)}\right) = 0.$$

Hence, $dm_\alpha \in \mathfrak{P}_0$ and

$$F_\alpha(z; dm_\alpha) = 1, \quad f_\alpha(z; dm_\alpha) = 0.$$

Besides, Proposition 2.4 implies that

$$f(z; dm_\alpha) = \overline{\zeta_0(\alpha)}, \quad F(z; dm_\alpha) = \frac{1 + \overline{\zeta_0(\alpha)} \zeta_\alpha(z)}{1 - \overline{\zeta_0(\alpha)} \zeta_\alpha(z)}.$$

This shows that the homeomorphism $\mathcal{B}_0\mathcal{C}_0$ between \mathfrak{P}_0 and \mathfrak{B} establishes a one to one correspondence between the set of measures $\{dm_\alpha : \alpha \in \mathbb{O}\}$ and the set of constant functions with values in \mathbb{D} .

The rest of constant S-functions are the constant unimodular ones, which the homeomorphism $\mathcal{B}_0\mathcal{C}_0$ puts in one to one correspondence with the set $\{\delta_\tau(t) = \delta(t - \tau) dt : \tau \in \partial\mathbb{O}\}$ of Dirac measures, since

$$F(z; \delta_\tau) = \frac{1 + \overline{\zeta_0(\tau)} \zeta_0(z)}{1 - \overline{\zeta_0(\tau)} \zeta_0(z)}, \quad f(z; \delta_\tau) = \overline{\zeta_0(\tau)}.$$

The fact that $f(z; dm_\alpha) = \overline{\zeta_0(\alpha)} \xrightarrow[\alpha \rightarrow \tau]{} f(z; \delta_\tau) = \overline{\zeta_0(\tau)}$ implies $dm_\alpha \xrightarrow[\alpha \rightarrow \tau]{*} \delta_\tau$. ■

The previous results deal only with the case of sequences $\alpha = (\alpha_n)$ compactly included in \mathbb{O} . If α is in \mathbb{O} but not compactly included there, a subsequence $(\alpha_{n_j})_j$ must exist such that $\operatorname{Lim}_j \alpha_{n_j} \subset \partial\mathbb{O}$, i.e., $|\zeta_0(\alpha_{n_j})| \xrightarrow{j} 1$. Concerning this situation we have the following strong convergence result.

Theorem 2.7. *If (α_k) is a sequence in \mathbb{O} such that $\operatorname{Lim} \alpha_k \subset \partial\mathbb{O}$, and $(d\mu^k)$ is a sequence in \mathfrak{P}_0 ,*

$$1 \notin \operatorname{Lim} f_{\alpha_k}(z; d\mu^k) \implies d\mu^k - dm_{\alpha_k} \xrightarrow{*} 0, \quad \operatorname{Lim} d\mu^k = \{\delta_\tau : \tau \in \operatorname{Lim} \alpha_k\}.$$

Proof. The relation between $g^k(z) = f_{\alpha_k}(z; d\mu^k)$ and $f^k(z) = f(z; d\mu^k)$ given by Proposition 2.4 can be written as

$$(1 - g^k) \left(f^k - \frac{1}{\zeta_0(\alpha_k)} \right) + \frac{1 - |\zeta_0(\alpha_k)|}{|\zeta_0(\alpha_k)|} \left(f^k + \frac{|\zeta_0(\alpha_k)|}{\zeta_0(\alpha_k)} \right) = 0.$$

The condition $\text{Lim } \alpha_k \subset \partial\mathbb{O}$ is equivalent to $|\zeta_0(\alpha_k)| \rightarrow 1$, so

$$(1 - g^k) \left(f^k - \overline{\zeta_0(\alpha_k)} \right) \Rightarrow 0.$$

The fact that $1 \notin \text{Lim } g^k$ forces $f^k - \overline{\zeta_0(\alpha_k)} \Rightarrow 0$, which, in view of Example 2.6, means that $f(z; d\mu^k) - f(z; dm_{\alpha_k}) \Rightarrow 0$. Then, the results follow from Theorem 2.5 and the last comment of Example 2.6. \square

2.2 The Nevalinna-Pick algorithm and the orthogonal rational functions

The Nevalinna-Pick algorithm comes from the fact that the transformation

$$f \rightarrow \frac{1}{\zeta_\alpha} \frac{f - f(\alpha)}{1 - \overline{f(\alpha)}f}$$

maps the interior $\mathfrak{B}^0 = \{f \in \mathcal{H}(\mathbb{O}) : |f(z)| < 1 \ \forall z \in \mathbb{O}\}$ of \mathfrak{B} on \mathfrak{B} for any $\alpha \in \mathbb{O}$. Given a sequence $\alpha = (\alpha_n)$ in \mathbb{O} , this algorithm associates with any $f \in \mathfrak{B}$ a finite or infinite sequence (f_n) in \mathfrak{B} defined by

$$\begin{aligned} f_0 &= f, \\ f_{n+1} &= \frac{1}{\zeta_{n+1}} \frac{f_n - \gamma_n}{1 - \overline{\gamma_n} f_n}, \quad \gamma_n = f_n(\alpha_{n+1}), \quad \zeta_n = \zeta_{\alpha_n}, \quad n \geq 0, \end{aligned} \quad (12)$$

so that the sequence terminates at f_N iff $f_N \in \mathfrak{B} \setminus \mathfrak{B}^0$, which holds iff f is, up to a unimodular factor, a finite Blaschke product $\zeta_{\beta_1} \zeta_{\beta_2} \cdots \zeta_{\beta_N}$ with $\beta_k \in \mathbb{O}$ for all k . We will say that (f_n) are the α -iterates of f and $\gamma = (\gamma_n)$ the α -parameters of f . Notice that (f_n, f_{n+1}, \dots) and $(\gamma_n, \gamma_{n+1}, \dots)$ are the iterates and parameters of f_n associated with the sequence $(\alpha_{n+1}, \alpha_{n+2}, \dots)$. From the relation between f_n and f_{n+1} we easily obtain

$$(1 - \overline{\gamma_n} f_n)(1 + \overline{\gamma_n} \zeta_{n+1} f_{n+1}) = 1 - |\gamma_n|^2, \quad (13)$$

an identity which will be useful later.

The maximum modulus principle implies that $\mathfrak{B} \setminus \mathfrak{B}^0$ is the set of constant unimodular functions. Therefore, $\gamma \in \mathfrak{S}$. Indeed, the map

$$\begin{array}{c} \mathcal{T}_\alpha: \mathfrak{B} \rightarrow \mathfrak{S} \\ f \mapsto \gamma \end{array} \quad (14)$$

is continuous for any sequence α in \mathbb{O} , as follows from the following theorem, which states a stronger result.

Theorem 2.8. *Let (α^k) be a sequence of sequences in \mathbb{O} , α a sequence in \mathbb{O} , (f^k) a sequence in \mathfrak{B} , and $f \in \mathfrak{B}$. Then,*

$$\alpha^k \rightarrow \alpha, f^k \rightrightarrows f \implies \gamma^k \rightarrow \gamma, f_n^k \xrightarrow{k} f_n \forall n,$$

where γ^k and $(f_n^k)_n$ are the α^k -parameters and α^k -iterates of f^k , while γ and (f_n) are the α -parameters and α -iterates of f , respectively.

Proof. Let $\gamma = (\gamma_n)_{n=0}^N$. We must prove that, for each $n \leq N$, f_n^k, γ_n^k exist for big enough k and $f_n^k \xrightarrow{k} f_n, \gamma_n^k \xrightarrow{k} \gamma_n$. Let us proceed by induction. First, $f_0^k = f^k$ exists for any k and $f_0^k \rightrightarrows f_0 = f$ from the hypothesis. Now, given $n \leq N$, suppose that f_n^k exist for big enough k and $f_n^k \xrightarrow{k} f_n$. Then, $\gamma_n^k = f_n^k(\alpha_{n+1}^k)$ exists for the same values of k and $\gamma_n^k \xrightarrow{k} \gamma_n = f_n(\alpha_{n+1})$. If $n = N$, there is nothing more to prove. Otherwise, $f_n \in \mathfrak{B}^0$, thus $f_n^k \in \mathfrak{B}^0$ for big enough k because $f_n^k \xrightarrow{k} f_n$. In consequence, f_{n+1}^k exists for such values of k . Moreover, denoting $\zeta_n^k = \zeta_{\alpha_n^k}$,

$$\begin{aligned} f_{n+1}^k - f_{n+1} &= \zeta_{n+1}^k f_{n+1}^k \left(\frac{1}{\zeta_{n+1}^k} - \frac{1}{\zeta_{n+1}} \right) + \frac{1}{\zeta_{n+1}} (\zeta_{n+1}^k f_{n+1}^k - \zeta_{n+1} f_{n+1}) = \\ &= \frac{1}{\zeta_{n+1}} \left[(\zeta_{n+1} - \zeta_{n+1}^k) f_{n+1}^k + \right. \\ &\quad \left. + \frac{(f_n^k - f_n) + (\gamma_n - \gamma_n^k) + (\bar{\gamma}_n^k - \bar{\gamma}_n) f_n^k f_n + \gamma_n^k \bar{\gamma}_n f_n - \bar{\gamma}_n \gamma_n f_n^k}{(1 - \bar{\gamma}_n^k f_n^k)(1 - \bar{\gamma}_n f_n)} \right], \end{aligned}$$

thus,

$$|f_{n+1}^k - f_{n+1}| \leq \frac{1}{|\zeta_{n+1}|} \left[|\zeta_n^k - \zeta_n| + \frac{2|f_n^k - f_n| + 4|\gamma_n^k - \gamma_n|}{(1 - |\gamma_n|)(1 - |\gamma_n^k|)} \right],$$

proving that $f_{n+1}^k \xrightarrow{k} f_{n+1}$ in $\mathbb{O} \setminus \{\alpha_{n+1}\}$, hence, in \mathbb{O} . □

Summarizing, given $\alpha \in \mathbb{O}$ and an arbitrary sequence $\boldsymbol{\alpha} = (\alpha_n)$ in \mathbb{O} , we have the following chain

$$\mathfrak{P}_0 \xrightarrow{\mathcal{C}_\alpha} \mathfrak{C}_\alpha \xrightarrow{B_\alpha} \mathfrak{B} \xrightarrow{\mathcal{T}_\alpha} \mathfrak{S}$$

the first two maps being homeomorphisms and the last one being continuous. For the choice $\alpha = \alpha_0$, the above diagram can be closed to a commutative one. This result is a consequence of the relation between S -functions and orthogonal rational functions.

Given a measure $d\mu \in \mathfrak{P}_0$ and a sequence $\boldsymbol{\alpha} = (\alpha_n)$ in \mathbb{O} , we can consider the orthonormalization in $L^2(d\mu)$ of the Blaschke products (B_n) given by

$$\begin{aligned} B_0 &= 1 \\ B_n &= \zeta_1 \zeta_2 \cdots \zeta_n, \quad n \geq 1. \end{aligned}$$

The result are the so called orthogonal rational functions (Φ_n) associated with $d\mu$ and $\boldsymbol{\alpha}$. Under a suitable normalization, they satisfy the recurrence relation (see [7, Theorem 4.1.3])

$$\begin{aligned} \Phi_0 &= 1, \\ \begin{pmatrix} \Phi_n \\ \Phi_n^* \end{pmatrix} &= e_n \frac{\varpi_{n-1}}{\varpi_n} \begin{pmatrix} 1 & \bar{\Lambda}_n \\ \Lambda_n & 1 \end{pmatrix} \begin{pmatrix} z_n \bar{z}_{n-1} \zeta_{n-1} \Phi_{n-1} \\ \Phi_{n-1}^* \end{pmatrix}, \quad n \geq 1, \\ \Phi_n^* &= B_n \Phi_{n*}, \quad \Lambda_n \in \mathbb{D}, \quad e_n = \sqrt{\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1}{1-|\Lambda_n|^2}}, \end{aligned} \quad (15)$$

where, for convenience, when α_n is a subindex it is denoted by n . In what follows, when referring to orthogonal rational functions we will suppose that they are normalized so that (15) holds. For our purposes, a more appropriate form of the above recurrence is in terms of the functions $\hat{\Phi}_n = \bar{z}_n \Phi_n$ and the parameters $\lambda_n = -z_{n+1} \Lambda_{n+1}$, i.e.,

$$\begin{pmatrix} \hat{\Phi}_n \\ \Phi_n^* \end{pmatrix} = e_n \frac{\varpi_{n-1}}{\varpi_n} T_{n-1} \begin{pmatrix} \hat{\Phi}_{n-1} \\ \Phi_{n-1}^* \end{pmatrix}, \quad T_n = \begin{pmatrix} \zeta_n & -\bar{\lambda}_n \\ -\lambda_n \zeta_n & 1 \end{pmatrix}. \quad (16)$$

Notice that $\lambda_n \in \mathbb{D}$ is given by $\lambda_n = -z_{n+1} \overline{\Phi_{n+1}(\alpha_n) / \Phi_{n+1}^*(\alpha_n)}$ and

$$e_n = \sqrt{\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1}{1-|\lambda_{n-1}|^2}}. \quad (17)$$

When $d\mu$ has an infinite support, there exists an infinite sequence of orthogonal rational functions which generates an infinite sequence (λ_n) in \mathbb{D} . If, on the contrary, $d\mu$ is supported on a finite number $N + 1$ of points, only the first $N + 1$ orthogonal rational functions Φ_0, \dots, Φ_N exist because $L^2(d\mu)$ is $N + 1$ -dimensional. Nevertheless, there exists $\Phi_{N+1} \in \text{span}\{B_0, \dots, B_{N+1}\}$ and orthogonal to Φ_0, \dots, Φ_N , although it has $L^2(d\mu)$ -norm equal to zero. Φ_{N+1} satisfies a relation like (15) with some coefficient $e_{N+1} \neq 0$ and $\lambda_N \in \mathbb{T}$. In consequence, given a sequence α in \mathbb{O} , we can associate with any measure $d\mu$ a sequence $\lambda = (\lambda_n) \in \mathfrak{S}$, which terminates iff $d\mu$ is finitely supported, the number of points in the support being equal to the length of λ . λ will be called the α -parameters of $d\mu$. As follows from [7, Theorem 8.1.4], this establishes a surjective map

$$\begin{array}{ccc} \mathcal{S}_\alpha: & \mathfrak{P}_0 & \rightarrow \mathfrak{S} \\ & d\mu & \rightarrow \lambda \end{array} \quad (18)$$

which is certainly bijective when $B_n \rightrightarrows 0$. When B_n does not diverge to 0, different probability measures can have the same α -parameters.

Simultaneously, we can consider the S-function of $d\mu$ and the corresponding α -parameters γ . The key result is that $\lambda = \gamma$, a fact which is an immediate consequence of [7, Corollary 6.5.2] (to fit with the notation there we must point out a misprint in formulas (6.27), (6.29) and (6.31), where z_n must be interchanged with \bar{z}_n ; then, $\lambda_n = L_{n+1}$ and $f_n = -\Gamma_n$). This result is the rational generalization of Geronimus' theorem for the orthogonal polynomials on the unit circle (see [9, 10, 11]). Therefore, for any sequence $\alpha = (\alpha_n)$ in \mathbb{O} , we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{P}_0 & \xrightarrow{\mathcal{C}_0} & \mathfrak{C}_0 \\ \mathcal{S}_\alpha \downarrow & & \downarrow \mathcal{B}_0 \\ \mathfrak{S} & \xleftarrow{\mathcal{T}_\alpha} & \mathfrak{B} \end{array}$$

with $\mathcal{C}_0, \mathcal{B}_0$ homeomorphisms and $\mathcal{T}_\alpha, \mathcal{S}_\alpha$ continuous and surjective. \mathcal{S}_α and \mathcal{T}_α are bijective when

$$B_n \rightrightarrows 0 \iff \begin{cases} \sum (1 - |\alpha_n|) = \infty, \\ \sum \frac{\text{Im } \alpha_n}{1 + |\alpha_n|^2} = \infty, \end{cases} \quad (19)$$

which means that not all the sequence α can approach to $\partial\mathbb{O}$ very quickly. If, on the contrary, B_n does not diverge to 0, different S-functions can have

the same α -parameters γ , and this occurs iff γ does not determine a unique probability measure. When this happens, we will say that we are in the indeterminate case.

When $B_n \rightrightarrows 0$, \mathcal{S}_α and \mathcal{T}_α are homeomorphisms, as can be deduced from Theorem 2.8 and the following important result.

Theorem 2.9. *Let (α^k) be a sequence of sequences in \mathbb{O} , α a sequence in \mathbb{O} with associated Blaschke products (B_n) , (f^k) a sequence in \mathfrak{B} and $f \in \mathfrak{B}$. If γ^k are the α^k -parameters of f^k and γ are the α -parameters of f , the condition $B_n \rightrightarrows 0$ ensures that*

$$\alpha^k \rightarrow \alpha, \gamma^k \rightarrow \gamma \implies f^k \rightrightarrows f.$$

Proof. Let $\tilde{f} \in \text{Lim } f^k$, i.e., $f^{k_j} \xrightarrow{j} \tilde{f}$ for some subsequence $(f^{k_j})_j$. Then, $\tilde{f} \in \mathfrak{B}$ and we can consider its α -parameters $\tilde{\gamma}$. From Theorem 2.8, $\gamma^{k_j} \xrightarrow{j} \tilde{\gamma}$, thus $\tilde{\gamma} = \gamma$. Hence, $B_n \rightrightarrows 0$ ensures that $\tilde{f} = f$. \square

In Theorems 2.8 and 2.9, if $\gamma = (\gamma_k)_{k=0}^N$, $N < \infty$, then the convergence condition $\alpha^k \rightarrow \alpha$ can be reduced to $\alpha_n^k \xrightarrow{k} \alpha_n$ for $n \leq N + 1$.

Example 2.10. From example 2.6, $f(z; dm_\alpha) = \overline{\zeta_0(\alpha)}$ for each $\alpha \in \mathbb{O}$ and $f(z; \delta_\tau) = \overline{\zeta_0(\tau)}$ for any $\tau \in \partial\mathbb{O}$. So, no matter the sequence α in \mathbb{O} ,

$$\mathcal{S}_\alpha(dm_\alpha) = (\overline{\zeta_0(\alpha)}, 0, 0, \dots), \quad \mathcal{S}_\alpha(\delta_\tau) = (\overline{\zeta_0(\tau)}).$$

Consider a sequence (α^k) of sequences all in the same compact subset of \mathbb{O} , and a sequence $(d\mu^k)$ in \mathfrak{P}_0 . If $\gamma^k = (\gamma_n^k)_n = \mathcal{S}_{\alpha^k}(d\mu^k)$, then

$$\text{Lim } d\mu^k \subset \{dm_\alpha : \alpha \in \mathbb{O}\} \implies \limsup |\gamma_0^k| < 1, \quad \gamma_n^k \xrightarrow{k} 0 \quad \forall n \geq 1,$$

$$\text{Lim } d\mu^k \subset \{\delta_\tau : \tau \in \partial\mathbb{O}\} \implies |\gamma_0^k| \rightarrow 1.$$

Besides, the above relations are “iff” when the Blaschke product related to α diverges to 0. We will only prove the first right implication since the rest of them follow analogous arguments.

Let us assume that $\text{Lim } d\mu^k \subset \{dm_\alpha : \alpha \in \mathbb{O}\}$ and let $\gamma = (\gamma_n) \in \text{Lim } \gamma^k$. Then, $\gamma^{k_j} \xrightarrow{j} \gamma$ for some subsequence $(\gamma^{k_j})_j$. Without loss of generality we can suppose $d\mu^{k_j} \xrightarrow[j]{*} dm_\alpha$, $\alpha \in \mathbb{O}$, if necessary restricting the subsequence.

By a similar reason, we can assume that $\alpha^{k_j} \xrightarrow{j} \alpha$ with α a sequence in the

same compact subset of \mathbb{O} than all the α^k . From Theorems 2.5 and 2.8 we find that $\gamma^{k_j} \xrightarrow{j} \mathcal{S}_\alpha(dm_\alpha)$, thus $\gamma_0 = \overline{\zeta_0(\alpha)} \in \mathbb{D}$ and $\gamma_n = 0$ for $n \geq 1$. Hence, $\gamma_n^k \xrightarrow{k} 0$ for $n \geq 1$, and $\limsup |\gamma_0^k| < 1$ because $\text{Lim } \gamma_0^k$ is a compact subset of \mathbb{D} . \blacksquare

We finish this section showing that the relation between the α -S-function and the S-function of a measure leads to a connection between their α -iterates and, thus, between their α -parameters. This connection is a simple consequence of the following general results.

Lemma 2.11. *Let α be a sequence in \mathbb{O} , $f, \tilde{f} \in \mathfrak{B}$ and denote by $(f_n), (\tilde{f}_n)$ and $(\gamma_n), (\tilde{\gamma}_n)$ the related α -iterates and α -parameters respectively.*

$$1. \quad \tilde{f} = \lambda f, \quad \lambda \in \mathbb{T} \quad \Rightarrow \quad \tilde{f}_n = \lambda f_n, \quad \tilde{\gamma}_n = \lambda \gamma_n, \quad n \geq 0.$$

$$2. \quad \tilde{f} = \frac{f - w}{1 - \overline{w}f}, \quad w \in \mathbb{D} \quad \Rightarrow \quad \begin{cases} \tilde{f}_0 = \frac{f_0 - w}{1 - \overline{w}f_0}, & \tilde{\gamma}_0 = \frac{\gamma_0 - w}{1 - \overline{w}\gamma_0}, \\ \tilde{f}_n = \frac{1 - w\overline{\gamma}_0}{1 - \overline{w}\gamma_0} f_n, & \tilde{\gamma}_n = \frac{1 - w\overline{\gamma}_0}{1 - \overline{w}\gamma_0} \gamma_n, \end{cases} \quad n \geq 1.$$

Proof. The first item is trivial. For the second one, in view of the first result, it suffices to prove it for $n = 1$, which is just a matter of computation. \square

As a direct consequence of the previous result and Proposition 2.4 we find that the sequences of α -iterates and α -parameters of S-functions and α -S-functions are proportional up to the first element of the sequence.

Proposition 2.12. *Let $\alpha \in \mathbb{O}$ and consider the S-function f and the α -S-function f_α of a measure $d\mu \in \mathfrak{P}_0$. If, for some sequence α in \mathbb{O} , $(f_n), (f_{\alpha,n})$ and $(\gamma_n), (\gamma_{\alpha,n})$ are the α -iterates and α -parameters of f, f_α respectively,*

$$\begin{aligned} f_{\alpha,0} &= -\frac{\zeta_0(\alpha)}{|\zeta_0(\alpha)|} \frac{f_0 - \overline{\zeta_0(\alpha)}}{1 - \overline{\zeta_0(\alpha)}f_0}, & \gamma_{\alpha,0} &= -\frac{\zeta_0(\alpha)}{|\zeta_0(\alpha)|} \frac{\gamma_0 - \overline{\zeta_0(\alpha)}}{1 - \overline{\zeta_0(\alpha)}\gamma_0}, \\ f_{\alpha,n} &= -\frac{\zeta_0(\alpha)}{|\zeta_0(\alpha)|} \frac{1 - \overline{\zeta_0(\alpha)}\gamma_0}{1 - \overline{\zeta_0(\alpha)}\gamma_0} f_n, & \gamma_{\alpha,n} &= -\frac{\zeta_0(\alpha)}{|\zeta_0(\alpha)|} \frac{1 - \overline{\zeta_0(\alpha)}\gamma_0}{1 - \overline{\zeta_0(\alpha)}\gamma_0} \gamma_n, \end{aligned} \quad n \geq 1.$$

3 Wall rational functions

Consider a sequence $\alpha = (\alpha_n)$ in \mathbb{O} and a S-function f with α -parameters $\gamma = (\gamma_n)$. The inverse relation between the α -iterates (f_n) of f can be

written as $f_{n-1} = M(\alpha_n, \gamma_{n-1})f_n$, where

$$M(\alpha, \gamma)f = \frac{\zeta_\alpha f + \gamma}{1 + \bar{\gamma} \zeta_\alpha f} = \gamma + \frac{(1 - |\gamma|^2) \zeta_\alpha}{\bar{\gamma} \zeta_\alpha + \frac{1}{f}}. \quad (20)$$

The identity $f = M(\alpha_1, \gamma_0)M(\alpha_2, \gamma_1) \cdots M(\alpha_n, \gamma_{n-1})f_n$ shows that

$$\begin{aligned} f = \gamma_0 + \frac{(1 - |\gamma_0|^2) \zeta_1}{\bar{\gamma}_0 \zeta_1} + \frac{1}{\gamma_1 + \frac{(1 - |\gamma_1|^2) \zeta_2}{\bar{\gamma}_1 \zeta_2}} + \cdots \\ \cdots + \frac{1}{\gamma_{n-1} + \frac{(1 - |\gamma_{n-1}|^2) \zeta_n}{\bar{\gamma}_{n-1} \zeta_n}} + \frac{1}{f_n}. \end{aligned} \quad (21)$$

This provides a formal expansion of f as an α -dependent continued fraction

$$f \sim \gamma_0 + \frac{(1 - |\gamma_0|^2) \zeta_1}{\bar{\gamma}_0 \zeta_1} + \frac{1}{\gamma_1 + \frac{(1 - |\gamma_1|^2) \zeta_2}{\bar{\gamma}_1 \zeta_2}} + \cdots + \frac{1}{\gamma_{n-1} + \frac{(1 - |\gamma_{n-1}|^2) \zeta_n}{\bar{\gamma}_{n-1} \zeta_n}} + \cdots,$$

which will be called the α -continued fraction of f . Its $2n - 2$ and $2n - 1$ approximants will be denoted $f^{(n)}$ and $\tilde{f}^{(n)}$ respectively, i.e.,

$$\begin{aligned} f^{(1)} = \gamma_0, \quad \tilde{f}^{(1)} = \gamma_0 + \frac{(1 - |\gamma_0|^2) \zeta_1}{\bar{\gamma}_0 \zeta_1}, \quad f^{(2)} = \gamma_0 + \frac{(1 - |\gamma_0|^2) \zeta_1}{\bar{\gamma}_0 \zeta_1} + \frac{1}{\gamma_1}, \\ \tilde{f}^{(2)} = \gamma_0 + \frac{(1 - |\gamma_0|^2) \zeta_1}{\bar{\gamma}_0 \zeta_1} + \frac{1}{\gamma_1 + \frac{(1 - |\gamma_1|^2) \zeta_2}{\bar{\gamma}_1 \zeta_2}}, \quad \cdots \end{aligned}$$

Notice that substituting f_n or $1/f_n$ by 0 in (21) yields respectively $f^{(n)}$ or $\tilde{f}^{(n)}$ instead of f . When f has a finite sequence $\gamma = (\gamma_n)_{n=0}^N$ of α -parameters, the related continued fraction is finite too because $\gamma_N \in \mathbb{T}$. In such a case, only the approximants $f^{(1)}, \dots, f^{(N+1)}$ and $\tilde{f}^{(1)}, \dots, \tilde{f}^{(N)}$ exist, and $f = f^{(N+1)}$ because the formal expansion as a continued fraction becomes an equality since $f_N = \gamma_N$.

$M(\alpha, \gamma)$ transforms rational functions into rational functions, thus $f^{(n)}$ and $\tilde{f}^{(n)}$ are both rational functions. Moreover, if $\alpha \in \mathbb{O}$ and $\gamma \in \mathbb{D}$, $M(\alpha, \gamma)$ maps \mathfrak{B} on \mathfrak{B}^0 . Therefore, $f^{(n)} \in \mathfrak{B}^0$ for all n , except for the case $N = 0$ where $f^{(1)} = f \in \mathfrak{B} \setminus \mathfrak{B}^0$. Two principal questions arise: What can we say about the expression and properties of $f^{(n)}$ and $\tilde{f}^{(n)}$? Do they converge to f ? The first question will lead to the rational analogue of the Wall polynomials. As for the convergence of $f^{(n)}$, an immediate answer emerges from the Nevalinna-Pick homeomorphisms.

Theorem 3.1. *Let α be a sequence in \mathbb{O} with related Blaschke products (B_n) , $f \in \mathfrak{B}$ and $f^{(n)}$ the $2n - 2$ approximant of the associated α -continued fraction. If $\gamma = (\gamma_n)_{n=0}^N$ are the α -parameters of f , the α -parameters of $f^{(n)}$ are $\gamma^{(n)} = (\gamma_0, \dots, \gamma_{n-1}, 0, 0, \dots)$ for $n < N + 1$. When $N = \infty$, the limit points of $(f^{(n)})$ are S -functions with α -parameters γ . In particular,*

$$B_n \rightrightarrows 0 \implies f^{(n)} \rightrightarrows f.$$

Proof. If $\alpha \in \mathbb{O}$ and $\gamma \in \mathbb{D}$, $g = M(\alpha, \gamma)h \in \mathfrak{B}^0$ and satisfies $g(\alpha) = \gamma$ for any $h \in \mathfrak{B}$. Therefore, if $n < N + 1$, $\gamma_0, \dots, \gamma_{n-1} \in \mathbb{D}$ and the relation $f^{(n)} = M(\alpha_1, \gamma_0)M(\alpha_2, \gamma_1) \cdots M(\alpha_n, \gamma_{n-1})0$ shows that $\gamma_0 = f^{(n)}(\alpha_1)$ and $M(\alpha_1, \gamma_0)^{-1}f^{(n)} = M(\alpha_2, \gamma_1) \cdots M(\alpha_n, \gamma_{n-1})0$ is the first α -iterate of $f^{(n)}$. By induction, the first n α -parameters of $f^{(n)}$ are $\gamma_0, \dots, \gamma_{n-1}$ and the n -th α -iterate of $f^{(n)}$ is 0. Hence, the rest of the α -parameters of $f^{(n)}$ are null. If $N = \infty$, then $\gamma^{(n)} \rightarrow \gamma$. Thus, the continuity of \mathcal{T}_α implies that the limit points of $(f^{(n)})$ must be S -functions with α -parameters γ . The condition $B_n \rightrightarrows 0$ ensures that \mathcal{T}_α is a homeomorphism, hence $f^{(n)} \rightrightarrows f$. \square

Notice that $\gamma^{(n)}$ is the sequence of $\tilde{\alpha}$ -parameters of $f^{(n)}$ whenever $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n, \tilde{\alpha}_{n+1}, \tilde{\alpha}_{n+2}, \dots)$, no matter the choice of $\tilde{\alpha}_j \in \mathbb{O}$ for $j > n$.

To analyze the nature of the approximants $f^{(n)}$ and $\tilde{f}^{(n)}$ we start writing the relation $f_{n-1} = M(\alpha_n, \gamma_{n-1})f_n$ between the α -iterates of f in the way

$$\begin{pmatrix} f_{n-1} \\ 1 \end{pmatrix} \doteq \begin{pmatrix} \zeta_n & \gamma_{n-1} \\ \bar{\gamma}_{n-1}\zeta_n & 1 \end{pmatrix} \begin{pmatrix} f_n \\ 1 \end{pmatrix},$$

where the symbol \doteq means equality up to a non vanishing scalar factor. Therefore,

$$\begin{pmatrix} f \\ 1 \end{pmatrix} \doteq \begin{pmatrix} \zeta_1 & \gamma_0 \\ \bar{\gamma}_0\zeta_1 & 1 \end{pmatrix} \begin{pmatrix} \zeta_2 & \gamma_1 \\ \bar{\gamma}_1\zeta_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} \zeta_{n+1} & \gamma_n \\ \bar{\gamma}_n\zeta_{n+1} & 1 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ 1 \end{pmatrix}.$$

It is evident that

$$\begin{pmatrix} f \\ 1 \end{pmatrix} \doteq \begin{pmatrix} \zeta_{n+1}\tilde{S}_n & R_n \\ \zeta_{n+1}\tilde{R}_n & S_n \end{pmatrix} \begin{pmatrix} f_{n+1} \\ 1 \end{pmatrix},$$

$R_n, S_n, \tilde{R}_n, \tilde{S}_n$ being linear combinations of the first $n + 1$ Blaschke products B_0, B_1, \dots, B_n related to α , with coefficients depending only on the parameters $\gamma_0, \dots, \gamma_n$. Hence,

$$f = \frac{R_{n-1} + \tilde{S}_{n-1} \zeta_n f_n}{S_{n-1} + \tilde{R}_{n-1} \zeta_n f_n}, \quad (22)$$

which is a compact way of writing (21). As we mentioned before, substituting f_{n+1} or $1/f_{n+1}$ by 0 in (21), i.e. in (22), we get respectively $f^{(n)}$ or $\tilde{f}^{(n)}$ instead of f . Thus,

$$f^{(n)} = \frac{R_{n-1}}{S_{n-1}}, \quad \tilde{f}^{(n)} = \frac{\tilde{S}_{n-1}}{\tilde{R}_{n-1}}.$$

Besides, \tilde{R}_n and \tilde{S}_n can be expressed in terms of R_n and S_n . From the equality

$$\begin{pmatrix} \zeta_{n+1}\tilde{S}_n & R_n \\ \zeta_{n+1}\tilde{R}_n & S_n \end{pmatrix} = \begin{pmatrix} \zeta_n\tilde{S}_{n-1} & R_{n-1} \\ \zeta_n\tilde{R}_{n-1} & S_{n-1} \end{pmatrix} \begin{pmatrix} \zeta_{n+1} & \gamma_n \\ \bar{\gamma}_n\zeta_{n+1} & 1 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \tilde{S}_n & R_n \\ \tilde{R}_n & S_n \end{pmatrix} = \begin{pmatrix} \tilde{S}_{n-1} & R_{n-1} \\ \tilde{R}_{n-1} & S_{n-1} \end{pmatrix} \begin{pmatrix} \zeta_n & \gamma_n\zeta_n \\ \bar{\gamma}_n & 1 \end{pmatrix}, \quad (23)$$

which, together with the initial condition

$$\begin{pmatrix} \tilde{S}_0 & R_0 \\ \tilde{R}_0 & S_0 \end{pmatrix} = \begin{pmatrix} 1 & \gamma_0 \\ \bar{\gamma}_0 & 1 \end{pmatrix}, \quad (24)$$

permits us to prove by induction that $\tilde{R}_n = R_n^* = B_n R_{n*}$ and $\tilde{S}_n = S_n^* = B_n S_{n*}$. In consequence,

$$f = \frac{R_{n-1} + S_{n-1}^* \zeta_n f_n}{S_{n-1} + R_{n-1}^* \zeta_n f_n}, \quad f^{(n)} = \frac{R_{n-1}}{S_{n-1}}, \quad \tilde{f}^{(n)} = \frac{S_{n-1}^*}{R_{n-1}^*}. \quad (25)$$

where R_n and S_n are recursively defined by

$$\begin{cases} R_0 = \gamma_0, \\ S_0 = 1, \end{cases} \quad \begin{cases} R_n = R_{n-1} + \gamma_n \zeta_n S_{n-1}^*, \\ S_n = S_{n-1} + \gamma_n \zeta_n R_{n-1}^*, \end{cases} \quad n \geq 1. \quad (26)$$

We will call (R_n) and (S_n) the Wall rational functions associated with f and α . The number of Wall rational functions coincides with the number of α -parameters of f . The recurrence for R_n^* and S_n^*

$$\begin{cases} R_0^* = \bar{\gamma}_0, \\ S_0^* = 1, \end{cases} \quad \begin{cases} R_n^* = \zeta_n R_{n-1}^* + \bar{\gamma}_n S_{n-1}, \\ S_n^* = \zeta_n S_{n-1}^* + \bar{\gamma}_n R_{n-1}, \end{cases} \quad n \geq 1, \quad (27)$$

shows that S_n^* is a monic element of $\text{span}\{B_0, \dots, B_n\}$, i.e., the coefficient of B_n in the expansion of S_n^* as a linear combination of B_0, \dots, B_n is 1. We can also get from (23) the inverse recurrence

$$\begin{cases} R_{n-1} = (1 - |\gamma_n|^2)^{-1}(R_n - \gamma_n S_n^*), \\ S_{n-1} = (1 - |\gamma_n|^2)^{-1}(S_n - \gamma_n R_n^*), \end{cases} \quad 1 \leq n < N.$$

The expression of the approximants $f^{(n)}$ and $\tilde{f}^{(n)}$ in terms of the Wall rational functions implies that $\tilde{f}^{(n)} = 1/f_*^{(n)}$. Thus, we obtain the following result for the convergence of $\tilde{f}^{(n)}$ as a direct consequence of Theorem 3.1.

Theorem 3.2. *Let α be a sequence in \mathbb{O} with related Blaschke products (B_n) , $f \in \mathfrak{B}$ and $\tilde{f}^{(n)}$ the $2n-1$ approximant of the associated α -continued fraction. If f has an infinite sequence of α -parameters,*

$$B_n \rightrightarrows 0 \implies \tilde{f}^{(n)} \rightrightarrows 1/f_* \text{ in } \mathbb{O}^e \setminus \{z \in \mathbb{O}^e : f(\hat{z}) = 0\}.$$

The above result has a natural interpretation in view of the integral representation of any S-function f . Let $d\mu$ be the probability measure such that $f(z) = f(z; d\mu)$. The expression for the corresponding C-function $F(z) = F(z; d\mu)$ defines an analytic function, not only for $z \in \mathbb{O}$, but also for $z \in \mathbb{O}^e$. This permits us to extend the definition of f to the points of \mathbb{O}^e throughout $f = \frac{1}{\zeta_0} \frac{F-1}{F+1}$. It is direct to see that $F = -F_*$, thus $f = 1/f_*$. This means that the extended f is analytic at $z \in \mathbb{O}^e$ iff f has not a zero at $\hat{z} \in \mathbb{O}$. With such an extended f , Theorems 3.1 and 3.2 can be combined to saying that, when $B_n \rightrightarrows 0$, then $f^{(n)} \rightrightarrows f$ in \mathbb{O} , while $\tilde{f}^{(n)} \rightrightarrows f$ in $\{z \in \mathbb{O}^e : f \text{ is analytic at } z\}$.

We can state some general properties of the Wall rational functions.

Proposition 3.3. *Let α be a sequence in \mathbb{O} with Blaschke products (B_n) , $(R_n)_{n=0}^N$, $(S_n)_{n=0}^N$ the Wall rational functions associated with $f \in \mathfrak{B}$, $(f_n)_{n=0}^N$ the corresponding α -iterates and $(\gamma_n)_{n=0}^N$ the related α -parameters. Let us denote $\Upsilon_n = \prod_{k=0}^n (1 - |\gamma_k|^2)$.*

1. $S_n(z) \overline{S_n(w)} - R_n^*(z) \overline{R_n^*(w)} =$
 $= (1 - |\gamma_n|^2)(S_{n-1}(z) \overline{S_{n-1}(w)} - \zeta_n(z) \overline{\zeta_n(w)} R_{n-1}^*(z) \overline{R_{n-1}^*(w)}).$
2. $|S_n|^2 - |R_n^*|^2 \geq \Upsilon_n$ in $\overline{\mathbb{O}}$, the inequality being an equality in $\partial\mathbb{O}$. Furthermore, $\Upsilon_n^{-1}(|S_n|^2 - |R_n^*|^2)$ is a non-decreasing sequence in $\overline{\mathbb{O}}$.

3. $S_n S_n^* - R_n R_n^* = \Upsilon_n B_n$.
4. S_n does not vanish in $\overline{\mathbb{O}}$ and has no zeros in common with R_n , neither with R_n^* .
5. $\frac{R_n}{S_n}, \frac{R_n^*}{S_n}, \frac{\Upsilon_n}{S_n^2} \in \mathfrak{B}$. Moreover, $\left| \frac{R_n}{S_n} \right|, \left| \frac{R_n^*}{S_n} \right| < 1$ in $\overline{\mathbb{O}}$ if $n < N$ and $\frac{R_N}{S_N} = f, \frac{R_N^*}{S_N} = \overline{\gamma}_N$ if $N < \infty$.
6. $f - \frac{R_n}{S_n} = \frac{\Upsilon_n}{S_n^2} \frac{B_{n+1} f_{n+1}}{1 + \frac{R_n^*}{S_n} \zeta_{n+1} f_{n+1}}$.
7. $\left| f - \frac{R_n}{S_n} \right| \leq (1 + |\zeta_{n+1}|) |B_{n+1}|$.
8. $\begin{cases} S_n + R_n^* \zeta_{n+1} f_{n+1} = \prod_{k=0}^n (1 + \overline{\gamma}_k \zeta_{k+1} f_{k+1}), \\ R_n + S_n^* \zeta_{n+1} f_{n+1} = (\gamma_0 + \zeta_1 f_1) \prod_{k=1}^n (1 + \overline{\gamma}_k \zeta_{k+1} f_{k+1}). \end{cases}$
9. $\begin{cases} S_n f - R_n = B_{n+1} f_{n+1} \prod_{k=0}^n (1 - \overline{\gamma}_k f_k), \\ S_n^* - R_n^* f = B_n \prod_{k=0}^n (1 - \overline{\gamma}_k f_k). \end{cases}$

Proof. Property 1 is a direct consequence of (26) and (27). Evaluating it for $w = z$ gives $|S_n|^2 - |R_n^*|^2 = (1 - |\gamma_n|^2)(|S_{n-1}|^2 - |\zeta_n|^2 |R_{n-1}^*|^2)$. Thus, $|S_n|^2 - |R_n^*|^2 \geq (1 - |\gamma_n|^2)(|S_{n-1}|^2 - |R_{n-1}^*|^2)$ in $\overline{\mathbb{O}}$, with an equality in $\partial\mathbb{O}$. This proves 2 since $|S_0|^2 - |R_0^*|^2 = 1 - |\gamma_0|^2$.

Taking determinants in (23) and (24) we get $S_0 S_0^* - R_0 R_0^* = 1 - |\gamma_0|^2$ and $S_n S_n^* - R_n R_n^* = \zeta_n (1 - |\gamma_n|^2) (S_{n-1} S_{n-1}^* - R_{n-1} R_{n-1}^*)$, which proves 3. The last equality follows also from Property 1 for $w = \hat{z}$.

Property 2 implies that, for $n < N$, $|S_n|^2 \geq \Upsilon_n > 0$ and $|S_n| > |R_n^*|$ in $\overline{\mathbb{O}}$. Besides, if $N < \infty$, using (26) we get $|S_N| \geq |S_{N-1}| - |R_{N-1}^*| > 0$ in $\overline{\mathbb{O}}$. Hence, S_n does not vanish in $\overline{\mathbb{O}}$ for any n . Then, Property 3 shows that S_n has no common zeros with $R_n R_n^*$ because B_n only vanishes at $\alpha_1, \dots, \alpha_n \in \mathbb{O}$.

Since S_n does not vanish in $\overline{\mathbb{O}}$, R_n/S_n , R_n^*/S_n and Υ_n/S_n^2 are analytic in a neighbourhood of $\overline{\mathbb{O}}$. From 2, $|\Upsilon_n/S_n^2| \leq 1$ and $|R_n^*/S_n| < 1$ for $n < N$ in $\overline{\mathbb{O}}$, while (26) and (27) give $R_N^*/S_N = \overline{\gamma}_N \in \mathbb{T}$ when $N < \infty$. Also, we know that $R_n/S_n = f^{(n)} \in \mathfrak{B}^0$ if $n < N$ and $R_N/S_N = f^{(N+1)} = f \in \mathfrak{B}$ for $N < \infty$. Moreover, $|R_n/S_n| = |R_n^*/S_n| < 1$ in $\partial\mathbb{O}$ for $n < N$.

Property 6 is obtained using (25) to express $f - f^{(n+1)}$ and simplifying the result with Property 3.

To prove 7, write 6 in the way

$$f - \frac{R_n}{S_n} = \frac{\Upsilon_n}{S_n^2} \frac{B_{n+1}f_{n+1}}{1 - \left(\frac{R_n^*}{S_n} \zeta_{n+1}f_{n+1}\right)^2} \left(1 - \frac{R_n^*}{S_n} \zeta_{n+1}f_{n+1}\right).$$

Then, use 2 and 5 to get $|\Upsilon_n/S_n^2| \leq 1 - |R_n^*/S_n|^2 \leq 1 - \left|\frac{R_n^*}{S_n} \zeta_{n+1}f_{n+1}\right|^2$ and $\left|1 - \frac{R_n^*}{S_n} \zeta_{n+1}f_{n+1}\right| \leq 1 + |\zeta_{n+1}|$ in \mathbb{O} .

From (12), (26), (27) and the help of (13) we arrive at the identities $S_n + R_n^* \zeta_{n+1} f_{n+1} = (1 + \bar{\gamma}_n \zeta_{n+1} f_{n+1})(S_{n-1} + R_{n-1}^* \zeta_n f_n)$ and $R_n + S_n^* \zeta_{n+1} f_{n+1} = (1 + \bar{\gamma}_n \zeta_{n+1} f_{n+1})(R_{n-1} + S_{n-1}^* \zeta_n f_n)$. This, together with the initial conditions given in (26) and (27), proves 8 by induction.

Using Properties 3 and 8 in (25), and taking into account identity (13), we get 9. \square

Property 7 of the above proposition measures the rate of the convergence $R_n/S_n \Rightarrow f$ when $B_n \Rightarrow 0$. In the polynomial situation it yields $|f - R_n/S_n| \leq (1 + |z|)|z|^{n+1}$, which improves the usual bounds given in the literature for this case.

The recurrence for the Wall rational functions permits us to identify certain iterates of the S-function R_n^*/S_n .

Proposition 3.4. *Let (R_n) , (S_n) be the Wall rational functions associated with a sequence $\alpha = (\alpha_n)$ in \mathbb{O} and an S-function f with α -parameters $\gamma = (\gamma_n)$. If $\tilde{\alpha} = (\alpha_n, \alpha_{n-1}, \dots, \alpha_1, \alpha_0, \alpha_0, \alpha_0, \dots)$, then the $\tilde{\alpha}$ -iterates and $\tilde{\alpha}$ -parameters of R_n^*/S_n are respectively*

$$(R_n/S_n, R_{n-1}/S_{n-1}, \dots, R_0/S_0, 0, 0, \dots), \quad (\bar{\gamma}_n, \bar{\gamma}_{n-1}, \dots, \bar{\gamma}_0, 0, 0, \dots).$$

Proof. It is simply a consequence of the identity

$$\frac{R_n^*}{S_n} = \frac{\zeta_n R_{n-1}^*/S_{n-1} + \bar{\gamma}_n}{1 + \gamma_n \zeta_n R_{n-1}^*/S_{n-1}}, \quad (28)$$

which is obtained directly from (26) and (27). \square

Proposition 3.4 also works with $\tilde{\alpha} = (\alpha_n, \alpha_{n-1}, \dots, \alpha_1, \tilde{\alpha}_{n+1}, \tilde{\alpha}_{n+2}, \dots)$, where $\tilde{\alpha}_j$ are arbitrary points of \mathbb{O} for $j > n$. As an immediate consequence of the previous results and Theorems 2.8, 2.9, we have that, for any sequences α^k, α in \mathbb{O} and any S-functions f^k, f ,

$$\alpha^k \rightarrow \alpha, f^k \rightrightarrows f \implies \frac{R_n^k}{S_n^k} \xrightarrow{k} \frac{R_n}{S_n}, \frac{R_n^{k*}}{S_n^k} \xrightarrow{k} \frac{R_n^*}{S_n} \quad \forall n,$$

where $(R_n^k)_n, (S_n^k)_n$ are the Wall rational functions associated with f^k, α^k and $(R_n), (S_n)$ are the Wall rational functions associated with f, α . Indeed, a stronger result can be obtained.

Proposition 3.5. *Let α^k be a sequence of sequences in \mathbb{O} , α a sequence in \mathbb{O} , (f^k) a sequence in \mathfrak{B} and $f \in \mathfrak{B}$. If $(R_n^k)_n, (S_n^k)_n$ are the Wall rational functions associated with f^k, α^k and $(R_n), (S_n)$ are the Wall rational functions associated with f, α , then, for all n ,*

$$\alpha^k \rightarrow \alpha, f^k \rightrightarrows f \implies \begin{cases} R_n^k \xrightarrow{k} R_n, & R_n^{k*} \xrightarrow{k} R_n^*, \\ S_n^k \xrightarrow{k} S_n, & S_n^{k*} \xrightarrow{k} S_n^*, \end{cases} \quad \text{in } \mathbb{C} \setminus \{\hat{\alpha}_1, \dots, \hat{\alpha}_n\}.$$

Proof. In view of Theorem 2.8, $\alpha^k \rightarrow \alpha$ and $f^k \rightrightarrows f$ imply $\gamma^k \rightarrow \gamma$, where γ^k are the α^k -parameters of $d\mu^k$ and γ are the α -parameters of $d\mu$. Then, the proof follows by induction using (26), (27) and taking into account that $\zeta_{\alpha_n^k} \xrightarrow{k} \zeta_{\alpha_n}$ in $\mathbb{C} \setminus \{\hat{\alpha}_n\}$. \square

Given a sequence α in \mathbb{O} , the Wall rational functions $(R_n), (S_n)$ associated with an S-function f are related to the orthogonal rational functions (Φ_n) corresponding to the measure $d\mu$ such that $f(z) = f(z; d\mu)$. The relation also involves the so called second kind rational functions (Ψ_n) , defined by

$$\begin{aligned} \Psi_0(z) &= 1, \\ \Psi_n(z) &= \int D(t, z)(\Phi_n(t) - \Phi_n(z)) d\mu(t), \quad n \geq 1. \end{aligned}$$

(Ψ_n) are orthogonal rational functions associated with the same sequence α , but with respect to a measure with α -parameters opposed to those ones of $d\mu$ (see [7, Theorems 4.2.4 and 6.2.5]). Therefore, $(\Psi_n, -\Psi_n^*)$ satisfy the same recurrence (15) as (Φ_n, Φ_n^*) , but with a different initial condition.

Proposition 3.6. *Let α be a sequence in \mathbb{O} , $(R_n)_{n=0}^N$, $(S_n)_{n=0}^N$ the Wall rational functions related to $f \in \mathfrak{B}$ and $(\Phi_n)_{n=0}^N$, $(\Psi_n)_{n=0}^N$ the orthogonal and second kind rational functions for the measure $d\mu \in \mathfrak{P}_0$ such that $f(z) = f(z; d\mu)$. Denoting $\kappa_n = (\Upsilon_{n-1} \varpi_0(\alpha_0) / \varpi_n(\alpha_n))^{1/2}$, we have for $n < N + 1$,*

$$\begin{cases} R_{n-1} = \frac{\kappa_n}{2} \frac{\varpi_n}{\varpi_0^*} (\Psi_n^* - \Phi_n^*), \\ R_{n-1}^* = \frac{\kappa_n}{2z_n} \frac{\varpi_n}{\varpi_0} (\Psi_n - \Phi_n), \end{cases} \quad \begin{cases} S_{n-1} = \frac{\kappa_n}{2} \frac{\varpi_n}{\varpi_0} (\Psi_n^* + \Phi_n^*), \\ S_{n-1}^* = \frac{\kappa_n}{2z_n} \frac{\varpi_n}{\varpi_0^*} (\Psi_n + \Phi_n), \end{cases}$$

$$\begin{cases} \Phi_n = \frac{z_n}{\kappa_n} \frac{\varpi_0}{\varpi_n} (\zeta_0 S_{n-1}^* - R_{n-1}^*), \\ \Phi_n^* = \frac{1}{\kappa_n} \frac{\varpi_0}{\varpi_n} (S_{n-1} - \zeta_0 R_{n-1}), \end{cases} \quad \begin{cases} \Psi_n = \frac{z_n}{\kappa_n} \frac{\varpi_0}{\varpi_n} (\zeta_0 S_{n-1}^* + R_{n-1}^*), \\ \Psi_n^* = \frac{1}{\kappa_n} \frac{\varpi_0}{\varpi_n} (S_{n-1} + \zeta_0 R_{n-1}). \end{cases}$$

Proof. Let us denote $\hat{\Phi}_n = \bar{z}_n \Phi_n$, $\hat{\Psi}_n = \bar{z}_n \Psi_n$ and

$$\mathbb{W}_n = \begin{pmatrix} S_n^* & -R_n^* \\ -R_n & S_n \end{pmatrix}, \quad \mathbb{F}_n = \begin{pmatrix} \hat{\Phi}_n & \hat{\Psi}_n \\ \Phi_n^* & -\Psi_n^* \end{pmatrix}.$$

With this notation, the recurrences for the Wall rational functions and the orthogonal and second kind rational functions read as

$$\mathbb{W}_n = T_n \mathbb{W}_{n-1}, \quad \mathbb{F}_n = e_n \frac{\varpi_{n-1}}{\varpi_n} T_{n-1} \mathbb{F}_{n-1},$$

with T_n and e_n given in (16) and (17) respectively. Hence,

$$\begin{aligned} \mathbb{W}_n &= T_n \cdots T_1 \mathbb{W}_0 = T_n \cdots T_1 T_0 \begin{pmatrix} \zeta_0 & 0 \\ 0 & 1 \end{pmatrix}^{-1}, \\ \mathbb{F}_n &= \frac{1}{\kappa_n} \frac{\varpi_0}{\varpi_n} T_{n-1} \cdots T_1 T_0 \mathbb{F}_0 = T_n \cdots T_1 T_0 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

So,

$$\mathbb{F}_n = \frac{1}{\kappa_n} \frac{\varpi_0}{\varpi_n} \mathbb{W}_{n-1} \begin{pmatrix} \zeta_0 & \zeta_0 \\ 1 & -1 \end{pmatrix},$$

which gives the desired relations. \square

The above result allows us to identify the measure and C-function corresponding to the S-function $f^{(n)} = R_{n-1}/S_{n-1}$. The measures dm_α , $\alpha \in \mathbb{O}$, given in Example 2.6 play an important role in such an identification.

Proposition 3.7. *Let α be a sequence in \mathbb{O} , $(\Phi_n)_{n=0}^N, (\Psi_n)_{n=0}^N$ the orthogonal and second kind rational functions of $d\mu \in \mathfrak{P}_0$, $(R_n)_{n=0}^N, (S_n)_{n=0}^N$ the Wall rational functions related to $f(z; d\mu)$ and $\gamma = (\gamma_n)_{n=0}^N$ its α -parameters. Then, for $n < N + 1$, we have the correspondences*

$$\begin{array}{ccccccc} \mathfrak{P}_0 & \xrightarrow{C_0} & \mathfrak{C}_0 & \xrightarrow{B_0} & \mathfrak{B} & \xrightarrow{T_\alpha} & \mathfrak{S} \\ \frac{dm_{\alpha_n}}{|\Phi_n|^2} & \longrightarrow & \frac{\Psi_n^*}{\Phi_n^*} & \longrightarrow & \frac{R_{n-1}}{S_{n-1}} & \longrightarrow & (\gamma_0, \dots, \gamma_{n-1}, 0, 0, \dots) \end{array}$$

Proof. From Theorem 3.1 we know that $(\gamma_0, \dots, \gamma_{n-1}, 0, 0, \dots)$ are the α -parameters of $f^{(n)} = R_{n-1}/S_{n-1}$. Besides, Theorem 3.6 gives

$$\frac{\Psi_n^*}{\Phi_n^*} = \frac{1 + \zeta_0 R_{n-1}/S_{n-1}}{1 - \zeta_0 R_{n-1}/S_{n-1}},$$

which shows that Ψ_n^*/Φ_n^* is a C-function and $R_{n-1}/S_{n-1} = \mathcal{B}_0(\Psi_n^*/\Phi_n^*)$. Finally, the fact that $dm_{\alpha_n}/|\Phi_n|^2$ is a probability measure with C-function Ψ_n^*/Φ_n^* was proven in [7, Theorem 4.2.6]. \square

Given a sequence α in \mathbb{O} and a measure $d\mu \in \mathfrak{P}_0$ with orthogonal rational functions (Φ_n) , we will denote

$$d\mu^{(n)} = \frac{dm_{\alpha_n}}{|\Phi_n|^2}, \quad (29)$$

so that, according to the previous notation, if f is the S-function of $d\mu$, $\gamma = (\gamma_n)$ its α -parameters and (Ψ_n) the second kind rational functions,

$$\begin{aligned} F(z; d\mu^{(n)}) &= \frac{\Psi_n^*(z)}{\Phi_n^*(z)}, & f(z; d\mu^{(n)}) &= \frac{R_{n-1}(z)}{S_{n-1}(z)} = f^{(n)}(z), \\ \mathcal{S}_\alpha(d\mu^{(n)}) &= (\gamma_0, \dots, \gamma_{n-1}, 0, 0, \dots) = \gamma^{(n)}. \end{aligned}$$

Notice that, if $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n, \tilde{\alpha}_{n+1}, \tilde{\alpha}_{n+2}, \dots)$ with $\tilde{\alpha}_j$ arbitrary points of \mathbb{O} for $j > n$, then $\mathcal{S}_{\tilde{\alpha}}(d\mu^{(n)}) = \gamma^{(n)}$ too. Using recurrence (15) we see that the orthogonal rational functions associated with $d\mu^{(n)}$ and $\tilde{\alpha}$ are $(\Phi_0, \dots, \Phi_n, \tilde{\Phi}_{n+1}, \tilde{\Phi}_{n+2}, \dots)$ where, using the tilde to refer to the elements related to $\tilde{\alpha}$,

$$\tilde{\Phi}_j = \sqrt{\frac{\tilde{\varpi}_j(\tilde{\alpha}_j)}{\varpi_n(\alpha_n)} \frac{\varpi_n^*}{\tilde{\varpi}_j} \frac{\tilde{B}_{j-1}}{B_n}} \Phi_n, \quad j > n.$$

Remember that R_n/S_n and S_n^*/R_n^* are respectively the $2n$ and $2n+1$ approximants of the α -continued fraction for $f(z) = f(z; d\mu)$. In [7, Section 4.4] it is shown that, analogously, Ψ_n^*/Φ_n^* and $-\Psi_n/\Phi_n$ are respectively the $2n$ and $2n+1$ approximants of an α -dependent continued fraction expansion of $F(z; d\mu)$ given by

$$1 - \frac{2}{1 + \gamma_0 \zeta_0 + \frac{(1 - |\gamma_0|^2) \zeta_0}{\overline{\gamma}_0} + \gamma_1 \zeta_1 + \frac{(1 - |\gamma_1|^2) \zeta_1}{\overline{\gamma}_1} + \dots} \dots \frac{1}{+ \gamma_n \zeta_n + \frac{(1 - |\gamma_n|^2) \zeta_n}{\overline{\gamma}_n} + \dots},$$

so that the even and odd approximants converge to $F(z; d\mu)$ in \mathbb{O} and \mathbb{O}^e respectively when the Blaschke product related to α diverges to 0. Under this condition we also have $d\mu^{(n)} \xrightarrow{*} d\mu$.

4 Rational Khrushchev's formula

The preceding results allow us to obtain a rational analogue of Khrushchev's formula for the orthogonal polynomials on the unit circle (see [13, Theorems 2 and 3]). We will state first a weak version of it. In what follows, the α -iterates of $d\mu$ means the α -iterates of $f(z; d\mu)$.

Theorem 4.1. *Let $\alpha = (\alpha_n)$ be a sequence in \mathbb{O} , $d\mu \in \mathfrak{P}_0$ and (Φ_n) the related orthogonal rational functions. If $b_n = \overline{z}_n \Phi_n / \Phi_n^*$ and (f_n) are the α -iterates of $d\mu$, then*

$$\frac{|\Phi_n(t)|^2}{D_R(t, \alpha_n)} \mu'(t) = \frac{1 - |f_n(t)|^2}{|1 - \zeta_n(t) b_n(t) f_n(t)|^2}, \quad a.e. \ t \in \partial\mathbb{O}.$$

Proof. From (5) we find that

$$\mu' = \operatorname{Re} \left(\frac{1 + \zeta_0 f}{1 - \zeta_0 f} \right) = \frac{1 - |f|^2}{|1 - \zeta_0 f|^2}, \quad a.e. \text{ on } \partial\mathbb{O}.$$

(25) and Property 2 of Proposition 3.3 yield

$$1 - |f|^2 = \Upsilon_{n-1} \frac{1 - |f_n|^2}{|S_{n-1} + R_{n-1}^* \zeta_n f_n|^2}, \quad a.e. \text{ on } \partial\mathbb{O}.$$

Using again (25), together with the relations of Proposition 3.6, we obtain

$$1 - \zeta_0 f = \kappa_n \frac{\varpi_n}{\varpi_0} \frac{\Phi_n^* - \bar{z}_n \zeta_n \Phi_n f_n}{S_{n-1} + R_{n-1}^* \zeta_n f_n},$$

hence

$$|1 - \zeta_0 f|^2 = \Upsilon_{n-1} \frac{\varpi_0(\alpha_0)}{\varpi_n(\alpha_n)} \left| \frac{\varpi_n}{\varpi_0} \right|^2 \frac{|\Phi_n|^2 |1 - \zeta_n b_n f_n|^2}{|S_{n-1} + R_{n-1}^* \zeta_n f_n|^2}, \quad \text{a.e. on } \partial\mathbb{D}.$$

Combining the previous equalities and taking into account (9) we get the result. \square

Notice that the equality of Theorem 4.1 is trivial when $d\mu$ is finitely supported because then f_n is a finite Blaschke product.

The functions $b_n = \bar{z}_n \Phi_n / \Phi_n^*$ are finite Blaschke products because the zeros of Φ_n lie on \mathbb{D} . Concerning their iterates, we have the following result.

Proposition 4.2. *Let $b_n = \bar{z}_n \Phi_n / \Phi_n^*$, where (Φ_n) are the orthogonal rational functions associated with a sequence $\alpha = (\alpha_n)$ in \mathbb{D} and a measure $d\mu \in \mathfrak{P}_0$ with α -parameters $\gamma = (\gamma_n)$. If $\tilde{\alpha} = (\alpha_{n-1}, \alpha_{n-2}, \dots, \alpha_1, \alpha_0, \alpha_0, \alpha_0, \dots)$, then, the $\tilde{\alpha}$ -iterates and $\tilde{\alpha}$ -parameters of b_n are respectively*

$$(b_n, b_{n-1}, \dots, b_0), \quad (-\bar{\gamma}_{n-1}, -\bar{\gamma}_{n-2}, \dots, -\bar{\gamma}_0, 1).$$

Proof. It follows immediately from the identity

$$b_n = \frac{\zeta_{n-1} b_{n-1} - \bar{\gamma}_{n-1}}{1 - \gamma_{n-1} \zeta_{n-1} b_{n-1}}$$

obtained from recurrence (15). \square

The above result also holds if $\tilde{\alpha} = (\alpha_{n-1}, \alpha_{n-2}, \dots, \alpha_0, \tilde{\alpha}_{n+1}, \tilde{\alpha}_{n+2}, \dots)$, where $\tilde{\alpha}_j$ are arbitrary points of \mathbb{D} for $j > n$. Following Khrushchev's terminology, we will call (b_n) the sequence of inverse α -iterates of $f(z; d\mu)$ or, equivalently, of $d\mu$. From the above proposition and Theorems 2.8, 2.9, we easily get a convergence property for the inverse α^k -iterates of $d\mu^k$ when $\alpha^k \rightarrow \alpha$ and $d\mu^k \xrightarrow{*} d\mu$. Moreover, using the relations of Proposition 3.6, we obtain from Proposition 3.5 a similar convergence property for the orthogonal rational functions of $d\mu^k$. We summarize all these results.

Proposition 4.3. *Let (α^k) be a sequence of sequences in \mathbb{O} , $\alpha = (\alpha_n)$ a sequence in \mathbb{O} , $(d\mu^k)$ a sequence in \mathfrak{P}_0 and $d\mu \in \mathfrak{P}_0$. If (Φ_n^k) , (b_n^k) are the orthogonal rational functions and inverse iterates associated with $d\mu^k$, α^k , and (Φ_n) , (b_n) are the orthogonal rational functions and inverse iterates associated with $d\mu$, α , then, for all n ,*

$$\alpha^k \rightarrow \alpha, \quad d\mu^k \xrightarrow{*} d\mu \implies \begin{cases} b_n^k \xrightarrow{k} b_n, \\ \Phi_n^k \xrightarrow{k} \Phi_n, \quad \Phi_n^{k*} \xrightarrow{k} \Phi_n^* \quad \text{in } \mathbb{C} \setminus \{\hat{\alpha}_1, \dots, \hat{\alpha}_n\}. \end{cases}$$

Now we can prove the strong version of Khrushchev's formula for the orthogonal rational functions.

Theorem 4.4 (First form of the rational Khrushchev's formula). *Let $\alpha = (\alpha_n)$ be a sequence in \mathbb{O} , $d\mu \in \mathfrak{P}_0$ and (Φ_n) the related orthogonal rational functions. If (f_n) and (b_n) are respectively the α -iterates and inverse α -iterates of $d\mu$, then*

$$f_{\alpha_n}(z; |\Phi_n|^2 d\mu) = b_n(z) f_n(z).$$

Proof. Let us suppose first that $d\mu(t) = \mu'(t) dt$, that is, $d\mu$ is absolutely continuous. Taking into account that $\mathcal{B}_{\alpha_n} \mathcal{C}_{\alpha_n}$ is a bijection between \mathfrak{P}_0 and \mathfrak{B} , the fact that $b_n f_n \in \mathfrak{B}$ ensures that $b_n(z) f_n(z) = f_{\alpha_n}(z; d\sigma_n)$ for some $d\sigma_n \in \mathfrak{P}_0$. In other words,

$$\frac{1 + \zeta_n(z) b_n(z) f_n(z)}{1 - \zeta_n(z) b_n(z) f_n(z)} = F_{\alpha_n}(z; d\sigma_n).$$

From (5),

$$\operatorname{Re} F_{\alpha_n}(t; d\sigma_n) = \frac{\sigma'_n(t)}{D_R(t, \alpha_n)}, \quad \text{a.e. } t \in \partial\mathbb{O}.$$

On the other hand, Theorem 4.1 gives for a.e. $t \in \partial\mathbb{O}$

$$\operatorname{Re} \left(\frac{1 + \zeta_n(t) b_n(t) f_n(t)}{1 - \zeta_n(t) b_n(t) f_n(t)} \right) = \frac{1 - |f_n(t)|^2}{|1 - \zeta_n(t) b_n(t) f_n(t)|^2} = \frac{|\Phi_n(t)|^2}{D_R(t, \alpha_n)} \mu'(t).$$

In consequence, $|\Phi_n|^2 \mu' = \sigma'_n$ a.e. on $\partial\mathbb{O}$. Bearing in mind that $d\sigma_n$ and $d\mu$ are probability measures, the equality $\int \sigma'_n(t) dt = \int |\Phi_n(t)|^2 \mu'(t) dt = \int |\Phi_n(t)|^2 d\mu(t) = 1$ shows that $d\sigma_n$ is absolutely continuous and, thus, $d\sigma_n = |\Phi_n|^2 d\mu$. Hence, we conclude that $b_n(z) f_n(z) = f_{\alpha_n}(z; |\Phi_n|^2 d\mu)$.

Consider now an arbitrary measure $d\mu \in \mathfrak{P}_0$, but supported on an infinite subset of $\partial\mathbb{D}$. The elements that appear in the rational Khrushchev's formula only depend on the measure $d\mu$ and the parameters $\alpha_1, \dots, \alpha_n$, but they are independent of the rest of parameters α_j , $j > n$. Therefore, we can suppose without loss of generality that $B_k \Rightarrow 0$. The absolutely continuous measures $d\mu^{(k)} = dm_{\alpha_k}/|\Phi_k|^2$ have the same n -th orthogonal rational function as $d\mu$ for $k \geq n$, so

$$b_n(z)f_n^{(k)}(z) = f_{\alpha_n}(z; |\Phi_n|^2 d\mu^{(k)}), \quad k \geq n, \quad (30)$$

where $(f_n^{(k)})_n$ are the α -iterates of $d\mu^{(k)}$. We know that $d\mu^{(k)} \xrightarrow{*} d\mu$ and $f^{(k)} \Rightarrow f$ where $f^{(k)}$, f are the S-functions of $d\mu^{(k)}$, $d\mu$ respectively. Hence, $f_n^{(k)} \xrightarrow{k} f_n$ for all n due to Theorem 2.8. Taking the limit $k \rightarrow \infty$ in (30), bearing in mind the continuity of $\mathcal{B}_{\alpha_n}\mathcal{C}_{\alpha_n}$, we get Khrushchev's formula for $d\mu$.

Finally, suppose that $d\mu \in \mathfrak{P}_0$ is finitely supported. We can obtain $d\mu$ as a $*$ -weak limit of measures $d\mu^k \in \mathfrak{P}_0$ supported on an infinite subset of $\partial\mathbb{D}$, for instance, $d\mu^k = \frac{k}{k+1}d\mu + \frac{1}{k+1}dm$. Denoting with the superscript k the elements corresponding to the measure $d\mu^k$ and the sequence α , we have

$$b_n^k(z)f_n^k(z) = f_{\alpha_n}(z; |\Phi_n^k|^2 d\mu^k). \quad (31)$$

From the continuity of $\mathcal{B}_0\mathcal{C}_0$, Theorem 2.8 and Proposition 4.3 we find that $d\mu^k \xrightarrow[k]{*} d\mu$ implies that $\Phi_n^k \xrightarrow[k]{\Rightarrow} \Phi_n$ in $\mathbb{C} \setminus \{\hat{\alpha}_1, \dots, \hat{\alpha}_n\}$, $b_n^k \xrightarrow[k]{\Rightarrow} b_n$ and $f_n^k \xrightarrow[k]{\Rightarrow} f_n$. Hence, Khrushchev's formula for $d\mu$ is obtained from (31) when $k \rightarrow \infty$. \square

It could seem surprising that, in the case of a measure with a singular part, the validity of Khrushchev's formula for any sequence α is obtained supposing that the related Blaschke product diverges to 0. Indeed, it is possible to accommodate the proof of the theorem to a general sequence α . We simply consider for any fixed n the new sequence $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n, \alpha_0, \alpha_0, \alpha_0, \dots)$, so that the related Blaschke product diverges to 0. Denoting with a tilde the elements related to $d\mu$ and $\tilde{\alpha}$, we have that $\tilde{\Phi}_n = \Phi_n$ and $\tilde{f}_n = f_n$. The absolutely continuous measures $d\tilde{\mu}^{(k)} = dm_{\tilde{\alpha}_k}/|\tilde{\Phi}_k|^2$ $*$ -weak converge to $d\mu$, their n -th orthogonal rational functions with respect $\tilde{\alpha}$ coincide with Φ_n for $k \geq n$, and their n -th $\tilde{\alpha}$ -iterates \tilde{f}_n^k satisfy $\tilde{f}_n^k \xrightarrow[k]{\Rightarrow} \tilde{f}_n = f_n$. So, we get Khrushchev's formula through a limiting process similar to the one given in the proof of the theorem.

Proposition 2.4 provides an equivalent version of the strong Khrushchev's formula.

Corollary 4.5 (Second form of the rational Khrushchev's formula). *With the notation of Theorem 4.4,*

$$f(z; |\Phi_n|^2 d\mu) = -\frac{|\zeta_0(\alpha_n)|}{\zeta_0(\alpha_n)} \frac{b_n(z)f_n(z) - |\zeta_0(\alpha_n)|}{1 - |\zeta_0(\alpha_n)| b_n(z)f_n(z)}.$$

5 The indeterminate case

Proposition 3.7 shows that, in the indeterminate case, to find the limit points of the sequence of approximants (R_n/S_n) of an S-function $f(z) = f(z; d\mu)$ is equivalent to find the limit points of the sequence of approximants (Ψ_n^*/Φ_n^*) of the C-function $F(z) = F(z; d\mu)$ or, alternatively, to find the limit points of the sequence of measures $(d\mu^{(n)})$. Due to its complexity, the convergence problem of the α -continued fraction of f in the indeterminate case will not be completely addressed in this paper, but we will provide some partial results to understand the special features of this problem, which does not appear in the polynomial setting. This discussion will also serve to show an example of application of Khrushchev's formula, whose validity for any sequence α makes of it a invaluable tool for studying the indeterminate case.

Let

$$\mathfrak{M}_\alpha(\gamma) = \{d\mu \in \mathfrak{P}_0 : \mathcal{S}_\alpha(d\mu) = \gamma\}, \quad \alpha \in \mathbb{O}, \quad \gamma \in \mathfrak{S}.$$

The determinate case refers to the situation where $\mathfrak{M}_\alpha(\gamma)$ has only one measure, otherwise we are in the indeterminate case. The indeterminate case can happen only if γ is infinite, so, the measures of $\mathfrak{M}_\alpha(\gamma)$ are necessarily infinitely supported in such a situation. Given a sequence α in \mathbb{O} , and bearing in mind the equality $\gamma_n = -z_{n+1}\Lambda_n$, recurrence (15) establishes a bijective relation between infinite sequences γ in \mathbb{D} and infinite sequences of orthogonal rational functions. Hence, the indeterminate case corresponds to an infinite sequence of orthogonal rational functions shared by different measures or, in other words, to an indeterminate rational moment problem: different measures $d\mu \in \mathfrak{P}_0$ giving the same values of $\int B_n d\mu$ for all $n \in \mathbb{N}$.

The indeterminate rational moment problem was studied in [4, 5, 6, 7, 8], following the analysis given in [3] for the polynomial situation on the real

line. In [5] and [7, Chapter 10] it was proved that, given $\alpha = (\alpha_n)$ and $\gamma = (\gamma_n)$,

$$\Delta(z) = \{F(z; d\mu) : d\mu \in \mathfrak{M}_\alpha(\gamma)\}, \quad z \in \mathbb{O}_0 = \mathbb{O} \setminus \{\alpha_k\}_{k=0}^\infty,$$

is always a disk or a point, depending whether we are in the indeterminate case or not. If (Φ_n) , (Ψ_n) are the orthogonal and second kind rational functions associated with α and γ throughout a recurrence like (15), then $\Delta(z)$ is a limit of nested disks

$$\Delta_n(z) = \{s \in \mathbb{C} : |\Psi_n^*(z) - s\Phi_n^*(z)| \leq |\Psi_n(z) + s\Phi_n(z)|\}, \quad (32)$$

which have centers $c_n(z)$ and radius $r_n(z)$ given by

$$c_n = \frac{\Psi_n^* \overline{\Phi_n^*} + \Psi_n \overline{\Phi_n}}{|\Phi_n^*|^2 - |\Phi_n|^2}, \quad r_n = \frac{|\Psi_n^* \Phi_n + \Psi_n \Phi_n^*|}{|\Phi_n^*|^2 - |\Phi_n|^2} = 2 \left| \frac{\varpi_0 \varpi_0^*}{\varpi_0(\alpha_0) \varpi} \right| \frac{|B_{n-1}|}{\sum_{k=0}^{n-1} |\Phi_k|^2}, \quad (33)$$

where $\varpi(z) = \varpi_z(z)$ and (B_n) are the Blaschke products related to α .

Equivalently, making the substitution $s \rightarrow \frac{1+\zeta_0(z)s}{1-\zeta_0(z)s}$ in (32) and using Proposition 3.6, we find that

$$\tilde{\Delta}(z) = \{f(z; d\mu) : d\mu \in \mathfrak{M}_\alpha(\gamma)\}, \quad z \in \mathbb{O}_0,$$

is always a disk or a point, depending whether we are in the indeterminate case or not, and $\tilde{\Delta}(z)$ is a limit of nested disks

$$\tilde{\Delta}_n(z) = \{s \in \mathbb{C} : |R_n(z) - sS_n(z)| \leq |S_n^*(z) - sR_n^*(z)|\}, \quad (34)$$

with centers $\tilde{c}_n(z)$ and radius $\tilde{r}_n(z)$ given by

$$\tilde{c}_n = \frac{R_n \overline{S_n} - S_n^* \overline{R_n^*}}{|S_n|^2 - |R_n^*|^2}, \quad \tilde{r}_n = \frac{|S_n S_n^* - R_n R_n^*|}{|S_n|^2 - |R_n^*|^2} = \frac{|B_n|}{\Upsilon_n^{-1}(|S_n|^2 - |R_n^*|^2)}, \quad (35)$$

where (R_n) , (S_n) are the Wall rational functions related to the sequences α and γ by recurrences (26) and (27).

The determinate case corresponds to the situation where Δ , or equivalently $\tilde{\Delta}$, is a point in \mathbb{O}_0 . In view of the expressions for r_n and \tilde{r}_n , this occurs iff B_n or $\sum_n |\Phi_n|^2$ diverge in \mathbb{O}_0 (to 0 and ∞ respectively), that is, iff B_n or $\Upsilon_n^{-1}(|S_n|^2 - |R_n^*|^2)$ diverge in \mathbb{O}_0 (to 0 and ∞ respectively). Therefore, the results of the previous sections that hold under the divergence of B_n , also

hold under the divergence of $\sum_n |\Phi_n|^2$, or equivalently $\Upsilon_n^{-1}(|S_n|^2 - |R_n^*|^2)$. For instance, these conditions ensure the convergence of (R_n/S_n) , (Ψ_n^*/Φ_n^*) and $(d\mu^{(n)})$.

On the contrary, in the indeterminate case, Δ and $\tilde{\Delta}$ are disks in \mathbb{O}_0 . In this situation B_n necessarily converges, thus $\zeta_n \rightarrow 1$ and $\text{Lim } \alpha_n \subseteq \partial\mathbb{O}$. As for the approximants of the continued fractions, we only know that $\text{Lim } (\Psi_n^*(z)/\Phi_n^*(z)) \subset \Delta(z)$ and $\text{Lim } (R_n(z)/S_n(z)) \subset \tilde{\Delta}(z)$ for any $z \in \mathbb{O}_0$. However, as we will see, we can say something more about the limit points of (Ψ_n^*/Φ_n^*) and (R_n/S_n) depending on the indeterminate moment problem at hand. Concerning the possibility of being in the indeterminate case for a given sequence $\gamma \in \mathfrak{S}$, we have the following result.

Lemma 5.1. *For any infinite sequence $\gamma \in \mathfrak{S}$ there exist infinitely many sequences α in \mathbb{O} such that $\mathfrak{M}_\alpha(\gamma)$ has more than one measure.*

Proof. Let $\gamma \in \mathfrak{S}$ be infinite. We will find sequences α in \mathbb{O} such that B_n and $\sum_n |\Phi_n|^2$ converge in \mathbb{O} . There, the inequality

$$|\Phi_n^*|^2 \leq \frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1 + |\gamma_{n-1}|}{1 - |\gamma_{n-1}|} \left| \frac{\varpi_{n-1}}{\varpi_n} \right|^2 |\Phi_{n-1}^*|^2,$$

obtained from (15), proves that

$$|\Phi_n|^2 \leq |\Phi_n^*|^2 \leq \left| \frac{\varpi_0}{\varpi_n} \right|^2 \frac{\varpi_n(\alpha_n)}{\varpi_0(\alpha_0)} \prod_{k=0}^{n-1} \frac{1 + |\gamma_k|}{1 - |\gamma_k|}. \quad (36)$$

Taking into account that

$$|\varpi_n(z)| \geq \begin{cases} 1 - |z|, \\ \text{Im } z, \end{cases} \quad z \in \mathbb{O},$$

(36) shows that the convergence of $\sum_n |\Phi_n|^2$ is a consequence of the convergence of $\sum_n \frac{\varpi_n(\alpha_n)}{\varpi_0(\alpha_0)} \prod_{k=0}^{n-1} \frac{1 + |\gamma_k|}{1 - |\gamma_k|}$. This last condition also implies the convergence of $\sum_n \frac{\varpi_n(\alpha_n)}{\varpi_0(\alpha_0)}$, which, bearing in mind (19), gives the convergence of B_n too. Therefore, it suffices to choose α such that $\sum_n \frac{\varpi_n(\alpha_n)}{\varpi_0(\alpha_0)} \prod_{k=0}^{n-1} \frac{1 + |\gamma_k|}{1 - |\gamma_k|}$ converges to ensure that α and γ correspond to the indeterminate case. \square

The fact that we are in the indeterminate case does not necessarily imply that (R_n/S_n) is non convergent. For instance, $R_n = 0$ and $S_n = 1$ if $\gamma_n = 0$

for all n . In such a case $R_n/S_n \Rightarrow 0$. However, $\Upsilon_n^{-1}(|S_n|^2 - |R_n^*|^2) = 1$ is always convergent, thus we are in the indeterminate case whenever B_n converges. Nevertheless, this is not the general situation. The following example shows that (R_n/S_n) can be actually non convergent in the indeterminate case. Notice that, from (35),

$$\frac{R_n}{S_n} - \tilde{c}_n = -\frac{B_n}{\Upsilon_n^{-1}(|S_n|^2 - |R_n^*|^2)} \frac{\overline{R_n^*}}{S_n}, \quad (37)$$

thus, in the indeterminate case, (R_n/S_n) converges iff $(\overline{R_n^*}/S_n)$ does so.

Example 5.2. Let $z \in \begin{cases} (0,1) \\ i(1,+\infty) \end{cases}$ be fixed. We will choose $\gamma_n \in (-1,1)$ and $\alpha_n \in \begin{cases} (-1,0) \\ i(0,1) \end{cases}$ so that $z \in \mathbb{O}_0$ and $d_n = \overline{R_n^*(z)}/S_n(z) = R_n^*(z)/S_n(z)$ defines a sequence in $(-1,1)$ given by

$$d_0 = \gamma_0; \quad d_n = \frac{\zeta_n(z) d_{n-1} + \gamma_n}{1 + \gamma_n \zeta_n(z) d_{n-1}}, \quad n \geq 1,$$

according to (28). Consider $\varepsilon_n \in (0,1)$ such that $\sum_n \varepsilon_n < \infty$. Fix $\gamma_0 \in (0,1)$ while, for each $n \geq 1$, define $\alpha_n \in \begin{cases} (-1,0) \\ i(0,1) \end{cases}$ by

$$\frac{\varpi_n(\alpha_n)}{\varpi_0(\alpha_0)} = \varepsilon_n \prod_{k=0}^{n-1} \frac{1 - |\gamma_k|}{1 + |\gamma_k|},$$

and choose $\gamma_n \in (-1,1)$ such that

$$\begin{cases} \max\{-\zeta_n(z) d_{n-1}, \gamma_0\} < \gamma_n < 1 & \text{if } n \text{ is even,} \\ -1 < \gamma_n < \min\{-\zeta_n(z) d_{n-1}, -\gamma_0\} & \text{if } n \text{ is odd.} \end{cases}$$

With this choice $d_n > 0$ for even n and $d_n < 0$ for odd n . Besides, $\sum_n \frac{\varpi_n(\alpha_n)}{\varpi_0(\alpha_0)} \prod_{k=0}^{n-1} \frac{1+|\gamma_k|}{1-|\gamma_k|} = \sum_n \varepsilon_n$ converges, thus we are in the indeterminate case, as follows from the proof of Lemma 5.1. If (d_n) converges, necessarily $d_n \rightarrow 0$. In such a case, $\gamma_n = (1 + \gamma_n \zeta_n(z) d_{n-1}) d_n - \zeta_n(z) d_{n-1}$ should converge to 0 too, but this is impossible because $|\gamma_n| \geq \gamma_0 > 0$. In consequence, $(\overline{R_n^*(z)}/S_n(z))$ does not converge, which means that $(R_n(z)/S_n(z))$ is non convergent because we are in the indeterminate case. ■

An interesting question is whether the limit points of (Ψ_n^*/Φ_n^*) are in the interior Δ^0 or the frontier $\partial\Delta$ of Δ , which is equivalent to a similar question concerning (R_n/S_n) and the interior $\tilde{\Delta}^0$ and frontier $\partial\tilde{\Delta}$ of $\tilde{\Delta}$. The reason is that the measures $d\mu \in \mathfrak{M}_\alpha(\gamma)$ have special features depending whether $F(z; d\mu)$ lies on $\Delta^0(z)$ or $\partial\Delta(z)$ (a fact which is independent of $z \in \mathbb{O}_0$, see [5] and [7, Chapter 10]). For example, the condition $F(z; d\mu) \in \partial\Delta(z)$ for $z \in \mathbb{O}_0$, which defines the so called N-extremal measures, characterizes the measures $d\mu \in \mathfrak{M}_\alpha(\gamma)$ such that (Φ_n) is a basis of $L^2(d\mu)$ (see [6] and [7, Chapter 10]). Moreover, if the limit points of α do not cover \mathbb{T} , the map

$$\begin{aligned} \mathfrak{M}_\alpha(\gamma) &\rightarrow \Delta(z) \\ d\mu &\longrightarrow F(z; d\mu) \end{aligned}$$

transforms only one measure into each point of $\partial\Delta$, while it transforms infinitely many measures into each point of Δ^0 (this is a consequence of the results in [8]; notice that this property does not appear correctly written in [7, Corollary 10.3.2]).

The modified approximant $(\Psi_n^* - \tau\Psi_n)/(\Phi_n^* + \tau\Phi_n)$ describes $\partial\Delta_n$ when τ runs over \mathbb{T} . Hence, given an arbitrary sequence (τ_n) in \mathbb{T} , the limit points of $(\Psi_n^* - \tau_n\Psi_n)/(\Phi_n^* + \tau_n\Phi_n)$ lie on $\partial\Delta$, i.e., they are C-functions of N-extremal measures, and any C-function of a N-extremal measure can be obtained as a limit of this kind of modified approximants (see [5] and [7, Chapter 10]). Using the relation between orthogonal rational functions and Wall rational functions we see that analogous results hold for the modified approximants $(R_n - \tau_n S_n^*)/(S_n - \tau_n R_n^*)$ and the S-functions of N-extremal measures. The aim of the next propositions is to know if something similar happens to the limit points of (Ψ_n^*/Φ_n^*) and (R_n/S_n) . This is equivalent to analyze the N-extremality of the limit points of the sequence of measures $(d\mu^{(n)})$.

Our first result concerning the limit points of (R_n/S_n) states that they lie on $\tilde{\Delta}^0$ when γ converges to zero quickly enough. In what follows, we will assume that we are in the indeterminate case.

Proposition 5.3. *If $\sum |\gamma_n| < \infty$, the limit points of $(R_n(z)/S_n(z))$ lie on $\tilde{\Delta}^0(z)$ for any $z \in \mathbb{O}_0$.*

Proof. Using (35) and (37) we find that

$$\left| \frac{R_n}{S_n} - \tilde{c}_n \right| = \left| \frac{R_n^*}{S_n^*} \right| \tilde{r}_n.$$

In consequence, in the indeterminate case, the limit points of $(R_n(z)/S_n(z))$ lie on $\Delta^0(z)$ for any $z \in \mathbb{O}_0$ iff $\text{Lim}(R_n^*/S_n)$ has no constant unimodular functions. This is also equivalent to $0 \notin \text{Lim}(\Upsilon_n/S_n^2)$, as follows from the identity

$$\tilde{r}_n = \frac{\Upsilon_n}{|S_n|^2} \frac{|B_n|}{1 - |R_n^*/S_n|^2},$$

obtained from (35).

From (26) and Proposition 3.3.5 we find that, in \mathbb{O} ,

$$|S_n| \leq \prod_{k=1}^n (1 + |\gamma_k|),$$

thus

$$\frac{\Upsilon_n}{|S_n|^2} \geq (1 - |\gamma_0|^2) \prod_{k=1}^n \frac{1 - |\gamma_k|}{1 + |\gamma_k|}.$$

Hence, $0 \notin \text{Lim}(\Upsilon_n/S_n^2)$ if $\prod_n \frac{1 - |\gamma_n|}{1 + |\gamma_n|}$ does not diverge to 0, i.e, if $\sum_n |\gamma_n|$ converges. \square

The above result does not hold in the general case, as the following proposition shows.

Proposition 5.4. *If $\limsup |\gamma_n| = 1$, at least one limit point of $(R_n(z)/S_n(z))$ lies on $\partial\hat{\Delta}(z)$ for $z \in \mathbb{O}_0$.*

Proof. Equivalently, we will prove a similar statement for $(\Psi_n^*(z)/\Phi_n^*(z))$. From (33) we find that

$$\left| \frac{\Psi_n^*}{\Phi_n^*} - c_n \right| = |b_n| r_n. \quad (38)$$

Therefore, in the indeterminate case, the limit points of $(\Psi_n^*(z)/\Phi_n^*(z))$ lie on $\Delta^0(z)$ for $z \in \mathbb{O}_0$ iff $\text{Lim } b_n$ has no unimodular constant functions.

Using (13) and Proposition 4.2 we get

$$(1 + \gamma_n b_{n+1})(1 - \gamma_n \zeta_n b_n) = 1 - |\gamma_n|^2.$$

So, if $\limsup |\gamma_n| = 1$, then $\liminf (1 - |b_{n+1}|)(1 - |b_n|) = 0$, which gives $\limsup |b_n| = 1$. Hence, $\text{Lim}(\Psi_n^*(z)/\Phi_n^*(z)) \not\subseteq \Delta^0(z)$ for $z \in \mathbb{O}_0$. \square

The next proposition gives a similar result to the previous one, but with a condition for the sequence α instead of γ . Remember that in the indeterminate case $\text{Lim } \alpha_n \subset \partial\mathbb{O}$.

Proposition 5.5. *If α has a limit point outside of $\text{supp}(d\mu)$, at least one limit point of $(R_n(z)/S_n(z))$ lies on $\partial\tilde{\Delta}(z)$ for $z \in \mathbb{O}_0$. Furthermore, if all the limit points of α are outside of $\text{supp}(d\mu)$, all the limit points of $(R_n(z)/S_n(z))$ lie on $\partial\tilde{\Delta}(z)$ for $z \in \mathbb{O}_0$.*

Proof. Applying Theorem 2.7 to α and the sequence of measures $(|\Phi_n|^2 d\mu)$, and taking into account Theorem 4.4, we get in the indeterminate case

$$\begin{aligned} 1 \notin \text{Lim}(b_n f_n) &\implies \text{Lim}(|\Phi_n|^2 d\mu) = \{\delta_\tau : \tau \in \text{Lim } \alpha_n\} \implies \\ &\implies \text{Lim } \alpha_n \subseteq \text{supp}(d\mu). \end{aligned}$$

Therefore, $\text{Lim } \alpha_n \not\subseteq \text{supp}(d\mu)$ implies $1 \in \text{Lim}(b_n f_n)$, so $\text{Lim } b_n$ contains unimodular constant functions. In such a case, following the arguments in the proof of Theorem 5.4 we find that $\text{Lim}(\Psi_n^*(z)/\Phi_n^*(z)) \not\subseteq \Delta^0(z)$ for $z \in \mathbb{O}_0$.

Suppose now that $b_n f_n$ does not converge to 1. Then, $1 \notin \text{Lim}_j(b_{n_j} f_{n_j})$ for a subsequence. Theorem 2.7 applied to $(\alpha_{n_j})_j$ and $(|\Phi_{n_j}|^2 d\mu)_j$ gives $\text{Lim}_j \alpha_{n_j} \subseteq \text{supp}(d\mu)$, so $\text{Lim } \alpha_n \cap \text{supp}(d\mu) \neq \emptyset$. We conclude that the condition $\text{Lim } \alpha_n \cap \text{supp}(d\mu) = \emptyset$ implies $b_n f_n \rightrightarrows 1$, thus $|b_n| \rightarrow 1$, which, bearing in mind (38), ensures that $\text{Lim}(\Psi_n^*(z)/\Phi_n^*(z)) \subseteq \partial\Delta(z)$ for $z \in \mathbb{O}_0$. \square

Theorems 5.3, 5.4 and 5.5 can be equivalently formulated as statements about the limit points of (Ψ_n^*/Φ_n^*) or, alternatively, about the N-extremality of the limit measures of $(d\mu^{(n)})$. For instance:

- 5.3. If $\sum |\gamma_n| < \infty$, none of the limit measures of $(d\mu^{(n)})$ is N-extremal.
- 5.4. If $\limsup |\gamma_n| = 1$, at least one limit measure of $(d\mu^{(n)})$ is N-extremal.
- 5.5. If α has a limit point outside of $\text{supp}(d\mu)$, at least one limit measure of $(d\mu^{(n)})$ is N-extremal. Furthermore, if all the limit points of α are outside of $\text{supp}(d\mu)$, all the limit measures of $(d\mu^{(n)})$ are N-extremal.

These results are enough to show the variety of possibilities for the convergence of (R_n/S_n) in the indeterminate case. Besides, Theorem 5.5 is obtained as an application of Khrushchev's formula, showing its interest for the analysis of problems related to the indeterminate case. Nevertheless, a complete study of the convergence of (R_n/S_n) in the indeterminate case should address the following problems:

1. A characterization of the cases where (R_n/S_n) is convergent, together with the description of the corresponding limit.

2. A complete description of the subset of $\tilde{\Delta}$ fulfilled by $\text{Lim}(R_n/S_n)$.
3. A characterization of the limit points of (R_n/S_n) .

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